Common fixed point theorems for self-mappings using Modified Ishikawa iteration scheme in Hilbert space

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Abstract: In this paper we have tried to establish common fixed point theorem for contractive mappings and prove common fixed-point theorems for pair of self-mappingsthrough rational expression using modified Ishikawa iteration scheme in Hilbert space.

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Key Words: Hilbert space, Contractive mapping, Common fixed-point and Ishikawa iteration scheme.

1. Introduction

Fixed point is the most important tool. The theory of fixed point is one of the most efficient areas for mathematics research. As for as we know about the application of fixed point theory which has developed much in several branches of mathematical science. The applicationof fixed point theory is in the branchesof economics, computer science, engineering, physical sciences, chemical science and mathematical science. The main objective of studies in the fixedpoint theory is to find solutions for the following equation which is commonly known as fixed point equation:

$$Tu = u \qquad (1)$$

Where, T be a self-mapping of a Hilbert space H and $u \in H$ .A wide variety of problems arise in all areas of physical, chemical and biological sciences, engineering, economics, and management can be modelled by linear or nonlinear equations of form

$$Fu = 0 \qquad (2)$$

Where, F is a linear or nonlinear equations. Equations of form (2) can be easily reformulated as a fixed-point equation of form (1). Since equation (1) has the same solution as the original equation (1), finding solutions of equation (1) leads to solutions of equation (2). To solve equations given by (1), two types of methods are normally used: (A) direct methods and (B) iterative methods. Due to various reasons, direct methods can be impractical or fail in solving equations (1), and thus iterative methods become a viable alternative. For this reason, the iterative approximation of fixed points has become one of the major and basic tools in the theory of equations.

Ciric [2] introduced the notation of generalized contraction mapping and proved fixed point theorems, Dass and Gupta [3] introduced an extension of Banach contraction principle through rational expression. The study of properties and applications of fixed points of various type of contractive mappings were obtained among authors by Rhoades, B. E. [13], [14], Magine, P. E. [7], Sayyed, S.A. and Badshah V.H. [16], [17],Naimpally, S.A. and Singh, K.L. [8], Nigam & dwivedi [9] and Pandhareand Waghmode [10].

Rhoades [15] introduced the convergence result of contractive operators using Mann and Ishikawa iterative schemes. Berinde [1] established the class of operators that is wider than the class contractive operators. It studied the convergence results of Ishikawa iteration process from this class of operators.Consequently, the literature of this highly dynamic research area abounds many iterative methods that have been introduced and developed by a wide audience of researchers to serve various purposes, viz., [1], [4], [5], [6], [11], [12], [15], [18], [19], [20] and [21 amongothers.Recently, Sharma, Badshah and Gupta [19] studied common fixed point in Hilbert space by Ishikawa iterations scheme.

We have introduced contractive operators on Hilbert space and prove that common fixed-point theorem on Hilbert space with modified Ishikawa iteration scheme.

2. Preliminaries

First, some important definitions and theorems, which are useful for main results.

Definition 2.1 Let K be a closed, convex subset of a Hilbert space H and let $S, T : K \to K$ be two self-mappings on a Hilbert space H. Suppose that the set $\{q : q \in T \,\&\, q \in S\}$ is common fixed point of $S \,\&\, T$.

We begin our exposition with an overview of various iterative methods

In 1953, W. R. Mann [6] introduced the following iterative scheme, for $u_0 \in K$ the sequence $\{u_n\}$ defined by

$$u_{n+1} = (1-\lambda_n)u_n + \lambda_n u_n$$

Where, $\{\lambda_n\}$ is sequence of non-negative numbers in $[0,1]$.

In 1974, S. Ishikawa [4] introduced the following iterative scheme, for $u_0 \in K$ the sequence $\{u_n\}$ defined by

$$u_{n+1} = (1-\lambda_n)u_n + \lambda_n T v_n$$
$$v_n = (1-\mu_n)u_n + \mu_n T u_n$$

Where, $\{\lambda_n\}$, $\{\mu_n\}$ are sequence of non-negative numbers in $[0,1]$ .and it is called Ishikawa iterative scheme.

Let the sequence $\{u_n\}$ be defined in accordance with the Mann iterates associated with T and S are given below:

For, $u_0 \in K$

$$\begin{aligned} u_{2n+1} &= (1-\lambda_n)u_{2n} + \lambda_n S u_{2n} \\ u_{2(n+1)} &= (1-\lambda_n)u_{2n+1} + \lambda_n T u_{2n+1} \end{aligned} \tag{3}$$

Where, $\{\lambda_n\}$ is sequence of non-negative numbers in $[0,1]$.

In 2014, Sharma A. K., Badshah V.H. and Gupta V.K. [19]introduced modified Ishikawa iteration scheme and the sequences $\{u_n\}$ and $\{v_n\}$ are defined by

For, $u_0 \in K$

$$\begin{aligned} v_{2n} &= (1-\mu_{2n})u_{2n} + \mu_{2n} S u_{2n} \\ u_{2n+1} &= (1-\lambda_{2n})u_{2n} + \lambda_{2n} T v_{2n} \\ v_{2n+1} &= (1-\mu_{2n+1})u_{2n+1} + \mu_{2n+1} S u_{2n+1} \\ u_{2n+2} &= (1-\lambda_{2n+1})u_{2n+1} + \lambda_{2n+1} T v_{2n+1} \end{aligned} \tag{4}$$

Where, $\{\lambda_{2n}\}$, $\{\mu_{2n}\}$ are sequence of non-negative numbers in $[0,1]$ for all n

$\lim_{n\to\infty} \mu_{2n} = 0$ and $\sum \lambda_{2n}\mu_{2n} = \infty$.

Here, we shall make the assumption that

(i) $0 \le \lambda_{2n} \le \mu_{2n} \le 1$, for all n

(ii) $\lim_{n\to\infty} \lambda_{2n} = \lambda > 0$

(iii) $\lim_{n\to\infty} \mu_{2n} = \lambda < 1$

Definition

Let K be a closed, convex subset of a Hilbert space H and let $S, T : K \to K$ be two self-mappings on a Hilbert space H. Then $T \,\&\, S$ are called non-expansive mapping if

$$\|Tu - Sv\| \le \|u - v\|,\ \forall u, v \in K$$

We know that Banach space is Hilbert space if and only if its norm satisfies the parallelogram law i.e., for every u, v

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

Which implies

$\|u+v\|^2 \le 2\|u\|^2 + 2\|v\|^2$

In 2014, Sharma, Badshah and Gupta [19] obtained by common fixed point theorem in Hilbert space with Ishikawa iteration scheme as follows

Theorem [19]: Let H be a Hilbert space and let K be a closed, convex subset of a Hilbert space H. Suppose that $S, T : K \to K$ be two self-mappings satisfying the condition

$$\|Tu - Sv\|^2 \le \left\{ \alpha + \beta \frac{\|u - Tu\|^2}{(1 + \|u - v\|^2)} \right\} \|v - Sv\|^2$$

where, $u, v \in K$ and $\alpha + \beta < \frac{1}{4}$. If there exists a point $u_0 \in K$, the sequence $\{u_n\}$ such that the I-iteration scheme for $S \& T$ defined by equation (4) converges to a point q, then q has common fixed point of $S \& T$.

We prove the result concerning the existence of common fixed point of pairs of mappings satisfying the contractive condition of the type

$$\|Tu - Sv\|^2 \le \frac{\lambda \|v - Sv\|^2 [1 - \|u - Tu\|^2]}{[1 - \|u - v\|^2]} + \mu [\|u - Sv\|^2 + \|u - Tv\|^2] + \gamma \|u - v\|^2$$

where, $u, v \in K$ and $2\lambda + 7\mu + 2\gamma < \frac{1}{2}$.

Main Results

Theorem: 3.1 Let K be a closed, convex subset of a Hilbert space H. Suppose that $S, T : K \to K$ be two self-mappings satisfying the condition

$$\|Tu - Sv\|^2 \le \frac{\lambda \|v - Sv\|^2 [1 - \|u - Tu\|^2]}{[1 - \|u - v\|^2]} + \mu[\|u - Sv\|^2 + \|u - Tv\|^2] + \gamma \|u - v\|^2 \qquad (5)$$

where, $u, v \in K$ and $2\lambda + 7\mu + 2\gamma < \frac{1}{2}$. If there exists a point $u_0 \in K$, the sequence $\{u_n\}$ such that the Ishikawa iteration scheme for $S \& T$ defined by equation (4)converges to a point q, then q has common fixed point of $S \& T$.

Proof: It follows from equation (4), we have

$$u_{2n+1} - u_{2n} = \alpha_{2n}(Sv_{2n} - u_{2n})$$

Since $u_{2n} \to q$, $\|u_{2n+1} - u_{2n}\| \to 0$. But $\{\alpha_{2n}\}$ is bounded away from zero, $\|Sv_{2n} - u_{2n}\| \to 0$. Therefore, $\|q - Sv_{2n}\| \to 0$. Since T and S satisfies (5), we have

$$\|Tu_{2n} - Sv_{2n}\|^2 \le \frac{\lambda \|v_{2n} - Sv_{2n}\|^2 [1 - \|u_{2n} - Tu_{2n}\|^2]}{[1 - \|u_{2n} - v_{2n}\|^2]} + \mu[\|u_{2n} - Sv_{2n}\|^2 + \|v_{2n} - Tu_{2n}\|^2] + \gamma \|u_{2n} - v_{2n}\|^2 \qquad (6)$$

Now,

$$\begin{aligned}
\|v_{2n} - u_{2n}\|^2 &= \|\beta_{2n} Tu_{2n} + (1 - \beta_{2n}) u_{2n} - u_{2n}\|^2 \\
&= \|\beta_{2n}(Tu_{2n} - u_{2n})\|^2 \\
&= \beta_{2n}^2 \|(Tu_{2n} - Sv_{2n}) + (Sv_{2n} - u_{2n})\|^2 \\
&\le 2\beta_{2n}^2 \|Tu_{2n} - Sv_{2n}\|^2 + 2\beta_{2n}^2 \|Sv_{2n} - u_{2n}\|^2 \\
&\le 2\|Tu_{2n} - Sv_{2n}\|^2 + 2\|Sv_{2n} - u_{2n}\|^2
\end{aligned}$$

$$\begin{aligned}\left\|v_{2n}-Sv_{2n}\right\|^2 &= \left\|\beta_{2n}Tu_{2n}+(1-\beta_{2n})u_{2n}-Sv_{2n}\right\|^2\\ &= \left\|\beta_{2n}(Tu_{2n}-Sv_{2n})+(1-\beta_{2n})(u_{2n}-Sv_{2n})\right\|^2\\ &\le 2\beta_{2n}^2\left\|Tu_{2n}-Sv_{2n}\right\|^2+2(1-\beta_{2n})^2\left\|u_{2n}-Sv_{2n}\right\|^2\\ &\le 2\left\|Tu_{2n}-Sv_{2n}\right\|^2+2\left\|u_{2n}-Sv_{2n}\right\|^2\end{aligned}$$

$$\begin{aligned}\left\|v_{2n}-Tu_{2n}\right\|^2 &= \left\|\beta_{2n}Tu_{2n}+(1-\beta_{2n})u_{2n}-Tu_{2n}\right\|^2\\ &= \left\|(1-\beta_{2n})(u_{2n}-Tu_{2n})\right\|^2\\ &= (1-\beta_{2n})^2\left\|u_{2n}-Tu_{2n}\right\|^2\\ &= (1-\beta_{2n})^2\left\|(u_{2n}-Sv_{2n})+(Sv_{2n}-Tu_{2n})\right\|^2\\ &\le 2(1-\beta_{2n})^2\left\|u_{2n}-Sv_{2n}\right\|^2+2(1-\beta_{2n})^2\left\|Sv_{2n}-Tu_{2n}\right\|^2\\ &\le 2\left\|u_{2n}-Sv_{2n}\right\|^2+2\left\|Sv_{2n}-Tu_{2n}\right\|^2\end{aligned}$$

From, equations (6), we have

$$\begin{aligned}\left\|Tu_{2n}-Sv_{2n}\right\|^2 &\le \frac{\lambda\{2\left\|Tu_{2n}-Sv_{2n}\right\|^2+2\left\|u_{2n}-Sv_{2n}\right\|^2\}[1-\left\|u_{2n}-Tu_{2n}\right\|^2]}{[1-\{2\left\|Tu_{2n}-Sv_{2n}\right\|^2+2\left\|Sv_{2n}-u_{2n}\right\|^2\}]}+\mu[\left\|u_{2n}-Sv_{2n}\right\|^2\\ &\quad+\{2\left\|u_{2n}-Sv_{2n}\right\|^2+2\left\|Sv_{2n}-Tu_{2n}\right\|^2\}]+\gamma\{2\left\|Tu_{2n}-Sv_{2n}\right\|^2+2\left\|Sv_{2n}-u_{2n}\right\|^2\}\\ &\le \frac{\lambda\{2\left\|Tu_{2n}-Sv_{2n}\right\|^2+2\left\|u_{2n}-Sv_{2n}\right\|^2\}[1-\{2\left\|Tu_{2n}-Sv_{2n}\right\|^2+2\left\|Sv_{2n}-u_{2n}\right\|^2\}]}{[1-\{2\left\|Tu_{2n}-Sv_{2n}\right\|^2+2\left\|Sv_{2n}-u_{2n}\right\|^2\}]}\\ &\quad+\mu[\left\|u_{2n}-Sv_{2n}\right\|^2+\{2\left\|u_{2n}-Sv_{2n}\right\|^2+2\left\|Sv_{2n}-Tu_{2n}\right\|^2\}]\\ &\quad+\gamma\{2\left\|Tu_{2n}-Sv_{2n}\right\|^2+2\left\|Sv_{2n}-u_{2n}\right\|^2\}\\ &= \lambda\{2\left\|Tu_{2n}-Sv_{2n}\right\|^2+2\left\|u_{2n}-Sv_{2n}\right\|^2\}+\mu[\left\|u_{2n}-Sv_{2n}\right\|^2\\ &\quad+\{2\left\|u_{2n}-Sv_{2n}\right\|^2+2\left\|Sv_{2n}-Tu_{2n}\right\|^2\}]+\gamma\{2\left\|Tu_{2n}-Sv_{2n}\right\|^2+2\left\|Sv_{2n}-u_{2n}\right\|^2\}\\ &= \left(2\lambda+2\mu+2\gamma\right)\left\|Tu_{2n}-Sv_{2n}\right\|^2+\left(2\lambda+3\mu+2\gamma\right)\left\|u_{2n}-Sv_{2n}\right\|^2\end{aligned}$$

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$$\|Tu_{2n} - Sv_{2n}\|^2 \le \frac{(2\lambda + 3\mu + 2\gamma)}{(1 - 2\lambda - 2\mu - 2\gamma)}\|u_{2n} - Sv_{2n}\|^2$$

Taking the limit as $n \to \infty$, since, $\|u_{2n} - Sv_{2n}\| \to 0$. Therefore, we have $\|Tu_{2n} - Sv_{2n}\| \to 0$.

It follows that,

$$\|u_{2n} - Tu_{2n}\|^2 \le 2\|u_{2n} - Sv_{2n}\|^2 + 2\|Sv_{2n} - Tu_{2n}\|^2 \to 0$$

And

$$\|q - Tu_{2n}\|^2 \le 2\|q - u_{2n}\|^2 + 2\|u_{2n} - Tu_{2n}\|^2 \to 0, \text{ as } n \to \infty$$

If u_{2n}, q satisfies (5) then, we have

$$\begin{aligned}\|Tu_{2n} - Sq\|^2 &\le \frac{\lambda\|q - Sq\|^2[1 - \|u_{2n} - Tu_{2n}\|^2]}{[1 - \|u_{2n} - q\|^2]} + \mu[\|u_{2n} - Sq\|^2 + \|q - Tu_{2n}\|^2] + \gamma\|u_{2n} - q\|^2 \\ &\le \frac{\lambda\{2\|q - Tu_{2n}\|^2 + 2\|Tu_{2n} - Sq\|^2\}[1 - \|u_{2n} - Tu_{2n}\|^2]}{[1 - \|u_{2n} - q\|^2]} \\ &\quad + \mu[\{2\|u_{2n} - Tu_{2n}\|^2 + 2\|Tu_{2n} - Sq\|^2\} + \|q - Tu_{2n}\|^2] + \gamma\|u_{2n} - q\|^2\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain $\|Tu_{2n} - Sq\| \to 0$.

Finally,

$$\begin{aligned}\|q - Sq\|^2 &= \|q - Tu_{2n} + Tu_{2n} - Sq\|^2 \\ &\le 2\|q - Tu_{2n}\|^2 + 2\|Tu_{2n} - Sq\|^2 \to \infty, \text{ as } n \to \infty\end{aligned}$$

Hence $q = Sq$.

Similarly, we can prove that $q = Tq$.

Hence q is the common fixed point of $T \& S$. This completes the proof

Corollary 3.1 Let K be a closed, convex subset of a Hilbert space H. Suppose that $S, T : K \to K$ be two self-mappings satisfying the condition

$$\|Tu - Sv\|^2 \leq \frac{\lambda \|v - Sv\|^2 [1 - \|u - Tu\|^2]}{[1 - \|u - v\|^2]}$$

where, $u, v \in K$ and $0 \leq \lambda < \frac{1}{4}$. If there exists a point $u_0 \in K$, the sequence $\{u_n\}$ such that the

Ishikawa iteration scheme for $S \& T$ defined by equation (4) converges to a point q, then q has common fixed point of $S \& T$.

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