An Algorithms to search Common Fixed Point of Nonexpansive mapping by iterative Technique

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Abstract: In this paper, we established a new iterative technique by the help of S-iteration technique and modified S-iteration technique. This technique is very fruitful for increasing the rate of convergence, after that we will prove the theorem of convergence of a sequence by our newly iterative technique, to determine a common fixed point in a Banach space under Nonexpansive mapping.
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1. Introdunction:

From the last few decades many researchers as well as mathematicians have been finding the approximation methods to determine fixed and common fixed point problems like optimization problem, variational inequalities etc. For the solving of those problems, they use various iterative schemes by various mapping such as Nonexpansive, Contractive, Contraction mapping etc.

The details of those problems can be seen in [1-12] studying by joining Mann and inertial extrapolation:

$$w_n = x_n + \alpha_n(x_n - x_{n-1}),$$
$$x_{n+1} = w_n + \beta_n\{s(w_n) - w_n\}, \quad \forall\, n \geq 1 \tag{1}$$

In this algorithms researcher provide the algorithms for the rate of convergence under the certain hypothesis. In above equation author also applied method to monotone convex feasibility problem, various fixed -point problems and monotone inclusion.

Inertial CQ algorithms and modified inertial Mann algorithms are introduced by Dong et. [13] by accelerated Mann algorithm with the inertial extrapolation. Let $T: H \to H$ be a nonexpansive mapping such that$Fix(T) \neq \emptyset$. Choose $\mu \,\epsilon\, (0,1), \lambda > 0$ and $x_0, x_1 \epsilon\, H$, arbitrarily and set $d_0 = (T(x_0) - x_0)/\lambda$, compute d_{n+1} and x_{n+1}as follows:

$w_n = x_n + \alpha_n(x_n - x_{n-1}),$

$$d_{n+1} = \frac{1}{\lambda}\,(T(w_n - w_n) + \beta_n d_n,$$
$$y_n = w_n + \lambda d_{n+1},$$
$$x_{n+1} = \mu\gamma_n w_n + (1 - \mu\gamma_n)\, y_n,\ \forall\, n \geq 1 \qquad (2)$$

Under some conditions they proved that the sequence $\{x_n\}$ generated by this algorithm converges weakly to a fixed point of T. Dong. et. Al. also researched an inertial CQ algorithms by combining the CQ- algorithm and the inertial extrapolation defined as follows: Let H be a Hilbert space and $T\colon H \rightarrow H$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$. Let $\{\alpha_n\}_{n=0}^{\infty} \subset [\alpha_1, \alpha_2]$, $\alpha_1 \in (-\infty, 0]$, $\alpha_2 \in (0, \infty]$, $\{\beta_n\}_{n=0}^{\infty} \subset [\beta_1, 1]$, $\beta_1 \in (0, 1)$. So $x_0, x_1 \in H$. Iterative sequence $\{x_n\}$ by the following iterative process:

$$w_n = x_n + \alpha_n (x_n - x_{n-1}),$$
$$y_n = (1 - \beta_n)\, w_n + \beta_n T w_n$$
$$C_n = \{z \in H : \|y_n - z\| \leq \|w_n - z\|\},$$
$$Q_n = \{z \in H : \langle x_n - z, x_n - x_0 \rangle \leq 0\}, \qquad (3)$$
$$x_{n+1} = P_{C_n \cap Q_n} x_0.$$

From (3) it's very clear that the sequence $\{x_n\}$ converges in the norm to$P_{Fix(T)}(x_0)$. In this property they also calculate some mathematical problem to illustrate the modified Mann algorithms and CQ algorithms by comparing the time with some previous techniques without extrapolation.

After the analysis on above said iterative techniques one renewed mathematician Suparatulatorn et.al. [16] Announced a new S- iterative technique as follows:

$x_0 \in C$ And
$$y_n = (1 - \beta_n) x_n + \beta_n S_1 x_n,$$
$$x_{n+1} = (1 - \alpha_n) S_1(x_n) + \alpha_n S_2(y_n). \qquad (4)$$

$n \geq 0$, Where C is a nonempty subset of a real Banach space, two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in (0,1) and here S_1 and S_2 are G-nonexpansive mappings. [16] also proved some theorems for weak and strong convergence under some certain conditions. They also determine common fixed point for convex Banach space by using mapping as G-nonexpansive. They also calculate some experimental problems by supporting of the idea that the sequence generated by modified S-iteration converges finer than the one generated by Ishikawa iteration.

Therefore, by aspiring from above, in this article we focus on joining of modified S-iteration technique and the inertial extrapolation to obtain new technique which accelerate the approximation of a fixed point of nonexpansive mapping in a Banach space defined as follows:

Let B be a Banach space and $T_1, T_2\, T_3 : B \rightarrow B$ be nonexpansive mapping such that $F = Fix(S_1) \cap Fix(S_2) \neq \emptyset$.

define as:

$$r_n = p_n + \gamma_n(p_n - p_{n-1})$$
$$q_n = (1-\beta_n)r_n + \beta_n T_1(r_n) \qquad (5)$$
$$p_{n+1} = (1-\alpha_n-\beta_n)\, T_1(r_n) + \alpha_n T_2(q_n) + \beta_n T_3(q_n), \forall\, n \geq 1.$$

Where $\{\alpha_n\}, \{\beta_n\}\, and\, \{\gamma_n\}$ satisfy:

(a) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\{\gamma_n\} \subset [0, \gamma], 0 \leq \gamma < 1, \{\alpha_n\}, \{\beta_n\} \subset [\delta, 1-\delta]$ for some $\delta > 0$.

(b) $\{T_i(w_n) - w_n\}$is bounded for $i = 1,2,3$. (*)

(c) $\{T_i(w_n) - y\}$ is bounded for $y \in F$ for $i = 1,2,3$.

We also show, under some assumptions, the weak and strong convergence of our newly iterative technique for determine common fixed point of T_1, T_2 and T_3.

2. Preliminaries

In this part we review some basic ideas and related lemmas by which we can able to show results. Here we start with the following identity which will be used at various places on this article.

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2, \qquad (**)$$
$\forall\, \alpha \in \mathbb{R}, x, y \in B$

Definition 2.1: A Banach space X is called to Opial property if whenever a sequence $\{x_n\}$ in X converges weakly to x, then

$$\lim_{n\to\infty} inf\|x_n - x\| \leq \lim_{n\to\infty} inf\|x_n - y\| \;\; \forall\, y \in X, y \neq x.$$

Definition 2.2 [17]**:** Let C be a nonempty closed convex subset of real uniformly convex Banach space X. The mapping S_1 and S_2 on C are said to satisfy condition B if there exists a nondecreasing function $f: [0,\infty) \rightarrow [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for $r > 0$ such that, for all $x \in C$,

$$max\{\|x - s_1(x)\|, \|x - s_2(x)\|\} \geq f(d(x, F)),$$

where $F = Fix(s_1) \cap Fix(s_2)$ and $Fix(s_i)$ is the set of fixed points of s_i for all $i = 1,2$.

Definition 2.3 [17]**:** Let C be a subset of a metric space *(X,d).* A mapping $S: C \rightarrow C$ is semicompact if for a sequence $\{x_n\}$ in C with $\log_{n\to\infty} d(x_n, s(x_n)) = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightarrow p \in C$.

Lemma 2.4 [18]**:** Let X be a uniformly convex Banach space, and α_n be a sequence in $[\delta, 1-\delta]$ for $\delta \in (0,1)$. Suppose that sequence $\{x_n\}$ and $\{y_n\}$ in X are such that

$lim \inf_{n\to\infty} \|x_n\| \le c$, $lim \inf_{n\to\infty} \|y_n\| \le c$, and $lim \inf_{n\to\infty} \|\alpha_n x_n + (1-\alpha_n) y_n\| = c$, for some $c \ge 0$. Then $lim \inf_{n\to\infty} \|x_n - y_n\| = 0$.

One eminent Mathematician Berinde compared the rate of convergence between the two iterative techniques by using the following definition.

Definition2.5: Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive numbers that converges to a and b, respectively. Let there exists

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = l.$$

- If $l = 0$, then it is said that the sequence $\{a_n\}$ converges to a finer than the sequence $\{b_n\}$ to b.
- If $0 < l < \infty$, then we say that the sequence $\{a_n\}$ and $\{b_n\}$ have the same rate of convergence.

Lemma 2.6[20]**:** Let *X* be a Banach space that has Opial's property and let $\{x_n\}$ be a sequence in *X*. Let $\{x_n\}$ be a sequence in *X*. Let *x, y* in *X* be such that $\lim_{n\to\infty}\|x_n - x\|$ *and* $\lim_{n\to\infty}\|x_n - y\|$ exist. If $\left\{x_{n_j}\right\}$ and $\left\{x_{n_k}\right\}$ are subsequences of $\{x_n\}$ that converges to *x* and *y*, respectively, then *x* = *y*.

Lemma 2.7[22]**:** Let *C* be a nonempty closed convex subset of real Hilbert space *H,* $T: C \to H$ a nonexpansive mapping. Let $\{x_n\}$ be a sequence in C and $x \in H$ such that $x_n \to x$ and $Tx_n \to x \to 0$ as n→ ∞. Then $x \in Fix(T)$.

Lemma 2.8[21]: Let $\{\psi_n\}, \{\delta_n\}$ and $\{\alpha_n\}$ be sequences in $[0,\infty]$ such that $\psi_{n+1} \le \psi_n + \alpha_n(\psi_n - \psi_{n-1}) + \delta_n$ for all $n \ge 1, \sum_{n=1}^{\infty} \delta_n < \infty$ and there exists a real number α with $0 \le \alpha_n < 1$ for all $n \ge 1$. Then the following hold:

(a) $\sum_{n\ge1}[\psi_n - \psi_{n-1}] < \infty$ where $[t] = max[t.0]$.
(b) $\exists\, \psi^* \in [0,\infty)$ such that $\log_{n\to\infty} \psi_n - \psi^*$.

3. Discussion on Results

In the part of this paper, we will discuss for the convergence of weak and strong sequences, which is generated by the new assumed algorithm to finding a fixed and common fixed point of three nonexpansive mappings.

Theorem 3.1: Let *X* be a uniformly convex Banach space. Let $Fix(T_1 \cap T_2 \cap T_3) = F$ and$y \in F$. Now we consider a sequence $\{x_n\}$ defined by (5). If (*) hold, then $\log_{n\to\infty}\|p_n - y\|$ exists.

Proof:

Firstly, we will use the nonexpansiveness of mappings and triangle inequality, so we have

$$\begin{aligned}\|q_n - y\| &= \|(1-\beta_n)\gamma_n + \beta_n T_1(\gamma_n) - y\| \\ &\leq (1-\beta_n)\|\,\gamma_n - y\,\| + \beta_n\|\,\gamma_n - y\,\| \\ &= \|\,\gamma_n - y\,\| \end{aligned} \quad (6)$$

$$\|p_{n+1} - y\| = \|(1-\alpha_n-\beta_n)T_1(r_n) + \alpha_n T_2(q_n) + \beta_n T_3(q_n) - y\|$$

$$\leq (1-\alpha_n-\beta_n)\|T_1(r_n) - y\| + \alpha_n\|T_2(q_n) - y\| + \beta_n\|T_3(q_n) - y\| \quad (7)$$

Now, nonexpansiveness of T_1, T_2, T_3 & (6), we have:

$$\begin{aligned}\|p_{n+1} - y\| &\leq (1-\alpha_n-\beta_n)\|T_1(r_n) - y\| + \alpha_n\|T_2(q_n) - y\| + \beta_n\|T_3(q_n) - y\| \\ &\leq (1-\alpha_n-\beta_n)\|r_n - y\| + \alpha_n\|q_n - y\| + \beta_n\|q_n - y\| \\ &\leq (1-\alpha_n-\beta_n)\|r_n - y\| + \alpha_n\|r_n - y\| + \beta_n\|r_n - y\| \\ &= \|r_n - y\| \end{aligned} \quad (8)$$

Here $\{r_n - y\}$ is bounded. So, by the using triangular inequality and condition of (*), we have

$$\begin{aligned}\|r_n - y\| &= \|r_n - T_1(r_n) + T_1(r_n) + T_2(r_n) - T_2(r_n) - y\| \\ &\leq \|T_1(r_n) - r_n\| + \|T_1(r_n) - T_2(r_n)\| + \|T_2(r_n) - y\| \\ &\leq K, \text{for some } K \in [0,\infty] \end{aligned}$$ i.e., $\{r_n - y\}$ is bounded.

So by (8), $\{x_n - y\}$ and $\{p_n - p_{n-1}\}$ are also bounded.

Now by identity (**), we have

$$\begin{aligned}\|r_n - y\|^2 &= \|p_n + \gamma_n(p_n - p_{n-1}) - y\|^2 \\ &= \|p_n(1+\gamma_n) - \gamma_n p_{n-1} - (1+\gamma_n)y + \gamma_n y\|^2 \\ &= (1+\gamma_n)\,\|(p_n - y)\|^2 - \gamma_n\|p_{n-1} - y\|^2 + \gamma_n(1+\gamma_n)\|p_n - p_{n-1}\|^2 \end{aligned} \quad (9)$$

$$\begin{aligned}\Rightarrow \|p_{n+1} - y\|^2 &\leq \|r_n - y\|^2 \\ &= (1+\gamma_n)\,\|(p_n - y)\|^2 - \|p_{n-1} - y\|^2 + \gamma_n(1+\gamma_n)\|p_n - p_{n-1}\|^2 \end{aligned} \quad (10)$$

Now put $\delta_n = \|p_n - y\|^2$, then (10) convert as:

$$\delta_n \leq \delta_n + \gamma_n(\delta_n - \delta_{n-1}) + \xi_n \quad (11)$$

Here

$$\xi_n = \gamma_n(1+\gamma_n)\|p_n - p_{n-1}\|^2$$

Here we are, observing by (* (a))

$$\begin{aligned}\sum_{n=1}^{\infty}\xi_n &= \sum_{n=1}^{\infty}\gamma_n(1+\gamma_n)\|p_n - p_{n-1}\|^2 \\ &\leq \sum_{n=1}^{\infty}\gamma_n(1-\gamma)(2K)^2 \\ &< \infty \end{aligned} \quad (12)$$

By lemma 2.8 (2),
There exists $\delta^* \in [0, \infty)$ s.t. $\lim_{n\to\infty} \delta_n = \delta^*$
$\Rightarrow \lim_{n\to\infty} \|p_n - y\|^2$ exist.
$\Longrightarrow \lim_{n\to\infty} \|p_n - y\|$ exist.

Theorem 3.2: Let X be a uniformly convex Banach space. Assume $y \in F = Fix(T_1) \cap Fix(T_2) \cap Fix\,(T_3)$. Let $\{p_n\}$ be a sequence defined by (5). If (a), (b) and (c) hold, then

$$\lim_{n\to\infty} \|p_n - T_1(p_n)\| = \lim_{n\to\infty} \|p_n - T_2(p_n)\| = \lim_{n\to\infty} \|p_n - T_3(p_n)\| = 0.$$

Proof: Here we will assume that $K = \lim_{n\to\infty} \|p_n - y\|$
so, by the nonexpansiveness property of T_1, T_2 & T_3 , we have

$$\|p_n - T_i(p_n)\| \le \|p_n - y\| + \|T_i(p_n) - y\| \le 2\|x_n - y\| \tag{13}$$

Now, we choose K = 0, then
$\|p_n - T_i(p_n)\| \to 0$, now assume that K > 0, note that
$\sum_{n=1}^{\infty} \gamma_n < \infty \Rightarrow \lim_{n\to\infty} \gamma_n = 0.$
It follows from (9)

$$\begin{aligned}\lim_{n\to\infty} \|r_n - y\|^2 &= \lim_{n\to\infty} ((1+\gamma_n)\|p_n - y\|^2 - \|p_{n-1} - y\|^2 \\ &\quad + \gamma_n(1+\gamma_n)\|p_n - p_{n-1}\|^2) \\ &= \lim_{n\to\infty} \|p_n - y\|^2 \\ &= K^2 \end{aligned} \tag{14}$$

$i.e. \lim_{n\to\infty} \|r_n - y\| = K$

so, from above we can get

$$\lim_{n\to\infty} sup\|q_n - y\| \le \lim_{n\to\infty} sup\|r_n - y\| = K$$

Now next we will show that
$\lim_{n\to\infty} sup\|q_n - y\| \ge K$, science T_1, T_2 & T_3 are nonexpansive, so by (**)

$$\begin{aligned}\|p_{n+1} - y\|^2 &= \|(1-\alpha_n-\beta_n)\,T_1(r_n) + \alpha_n T_2(q_n) + \beta_n T_n(q_n) - y\|^2 \\ &\le (1-\alpha_n-\beta_n)\|T_1(r_n) - y\|^2 + \alpha_n\|T_2(q_n) - y\|^2 + \beta_n\|T_3(q_n) - y\|^2 \\ &\le (1-\alpha_n-\beta_n)\|r_n - y\|^2 + (\alpha_n+\beta_n)\|q_n - y\|^2 \\ &\le (1-\alpha_n-\beta_n)[\|r_n\|^2 + (1-r_n)\|y\|^2 - \|r_n - y\|^2] \\ &\quad + (\alpha_n+\beta_n)[\|q_n\|^2 - (1-q_n)\|y\|^2 - \|q_n - y\|^2] \\ &\le (1-\alpha_n-\beta_n)\|r_n - y\|^2 + (\alpha_n+\beta_n)\|q_n - y\|^2 \end{aligned} \tag{15}$$

Now rearranging (15) and using (a) of (*), then

$$\|r_n - y\|^2 \le \|q_n - y\|^2 + \frac{1}{\alpha_n+\beta_n}(\|r_n - y\|^2 - \|q_{n+1} - y\|^2)$$

$$\leq \|q_n - y\|^2 + \frac{1}{\delta}(\|r_n - y\|^2 - \|q_{n+1} - y\|^2) \quad (16)$$

By (16) and (8), we can observe that
$\lim_{n\to\infty} inf\ \|q_n - y\|^2 \geq K^2 \Rightarrow \lim_{n\to\infty} inf\ \|q_n - y\| \geq K$ -

Since
$K \leq \lim_{n\to\infty} inf\ \|q_n - y\| \leq \lim_{n\to\infty} sup\ \|q_n - y\| \leq K$
$\Longrightarrow \lim_{n\to\infty}\|q_n - y\| \leq K$
Since

$$\lim_{n\to\infty} sup\ \|T_1(r_n) - y\| \leq \lim_{n\to\infty} sup\ \|r_n - y\| \leq K,$$
$$\lim_{n\to\infty} sup\ \|T_2(q_n) - y\| \leq \lim_{n\to\infty} sup\ \|q_n - y\| \leq K,$$

And

$$\lim_{n\to\infty} sup\ \|T_3(q_n) - y\| \leq \lim_{n\to\infty} sup\ \|q_n - y\| \leq K,$$

here $\lim_{n\to\infty}\|(1-\alpha_n - \beta_n)(r_n - y) + (\alpha_n + \beta_n)(r_n - y)\| \lim_{n\to\infty}\|r_n - y\| = K,$

$$\lim_{n\to\infty}\|(1-\beta_n)[T_1(r_n) - y) + \beta_n (T_2(r_n) - y)]\| = \lim_{n\to\infty}\|q_{n+1} - y\| = K,$$

And

$$\lim_{n\to\infty}\|(1-\alpha_n)[T_2(r_n) - y) + \alpha_n (T_3(r_n) - y)]\| = \lim_{n\to\infty}\|q_{n+1} - y\| = K,$$

then using (2.4)

$$\lim_{n\to\infty}\|T_1(r_n) - r_n\| = 0 \quad (17)$$
$$\lim_{n\to\infty}\|T_1(r_n) - T_2(q_n)\| = 0 \quad (18)$$

And

$$\lim_{n\to\infty}\|T_2(q_n) - T_3(p_{n+1})\| = 0 \quad (19)$$

However, we know that $q_n - r_n = \beta_n (T_1(r_n) - r_n)$ and $r_n - p_n = \gamma_n(p_n - p_{n-1})$
Which yield

$$\lim_{n\to\infty}\|q_n - r_n\| \geq 0$$
$$\lim_{n\to\infty} \beta_n \|(T_1(r_n) - r_n)\| = 0$$
$$\lim_{n\to\infty} \|(T_1(r_n) - r_n)\| \geq 0$$
$$= 0. \quad (20)$$

and

$$\lim_{n\to\infty}\|r_n - p_n\| = \lim_{n\to\infty} \gamma_n \|p_n - p_{n-1}\|$$
$$= 0 \quad (21)$$

So, by (b) from (*) $\gamma_n \to 0$.
We have

$$\lim_{n\to\infty}\|T_1(r_n) - p_n\| \leq \lim_{n\to\infty}\|T_1(r_n) - r_n\| + \lim_{n\to\infty} \gamma_n\|p_n - p_{n-1}\|$$
$$= 0. \quad (22)$$

Now, it follows that form (19), (20), (21) & (22) and nonexpansiveness of T_1, T_2 and T_3, we have

$$0 \le \lim_{n\to\infty} \|T_1(p_n) - p_n\|$$
$$0 \le \lim_{n\to\infty}\|T_1(p_n) - T_1(r_n)\| + \lim_{n\to\infty}\|T_1(r_n) - p_n\|$$
$$0 \le \lim_{n\to\infty}\|p_n - r_n\| + \lim_{n\to\infty}\|T_1(p_n) - r_n\|$$
$$= 0 \quad (23)$$

in similar manner, we can get

$$0 \le \lim_{n\to\infty}\|T_2(p_n) - p_n\|$$
$$= 0 \quad (24)$$

and

$$0 \le \lim_{n\to\infty}\|T_3(p_n) - p_n\|$$
$$= 0 \quad (25)$$

i.e. $\lim_{n\to\infty}\|T_1(p_n) - p_n\| = \lim_{n\to\infty}\|T_2(p_n) - p_n\| = \lim_{n\to\infty}\|T_3(p_n) - p_n\| = 0.$

Theorem 3.3: Let H be a Banach space having opial property. Let T_1, T_2 & T_3: $H \to H$ are three nonexpansive mapping with F = Fix(T_1) ∩ Fix(T_2) ∩ Fix(T_3) ≠ ∅. Then the sequence $\{x_n\}$ in (5) converges weakly to common fixed points of T_1, T_2, and T_3.

Proof: Let a point $y \in F$, then by theorem 3.1 $\lim_{n\to\infty}\|p_n - y\|$ exists. Hence $\{p_n\}$ is bounded. Now let $\{p_{n_i}\}$, $\{p_{n_j}\}$ and $\{p_{n_k}\}$ be the three subsequences of $\{p_n\}$, having limits are l_1, l_2 and l_3 respectively, then by the theorem 3.2 we have as follow
$\lim_{n\to\infty}\|p_{n_i} - T_t(p_{n_i})\| = \lim_{n\to\infty}\|p_{n_j} - T_t(p_{n_j})\| = \lim_{n\to\infty}\|p_{n_k} - T_t(p_{n_k})\| = 0$ for t = 1,2,3. By lemma 2.7, we have $T_i(l_1) = l_1$, $T_i(l_2) = l_2$ & $T_i(l_3) = l_3 \;\forall\, i = 1,2,3$.Now by theorem 3.1, again we have $\lim_{n\to\infty}\|p_n - l_1\|$, $\lim_{n\to\infty}\|p_n - l_2\|$ *and* $\lim_{n\to\infty}\|p_n - l_3\|$ are all exist and all the sequences $\{p_{n_i}\}$, $\{p_{n_j}\}$ and $\{p_{n_k}\}$ are converging to l_1, l_2 and l_3 respectively. By lemma (2.6), $l_1 = l_2 = l_3$.
Hence, $\{x_n\}$ converges weakly to a common fixed point in F.

Theorem 3.4: Let H be a uniform convex Banach space. Let T_1, T_2 & T_3: $H \to H$ are three nonexpansive mapping with F = Fix(T_1) ∩ Fix(T_2) ∩ Fix(T_3) ≠ ∅ and satisfy the condition B. Then the sequence $\{p_n\}$ in (5) converges strongly to a common fixed point of T_1, T_2 & T_3.

Proof: Here we assume that $y \in F$. so by (9), we find

$$\inf_{y\in F}\{\|p_{n+1} - y\|^2\} \le \inf_{y\in F}\{\|r_n - y\|^2\}$$
$$= \inf_{y\in F}\{(1+\gamma_n)\|p_n - y\|^2\} + \inf_{y\in F}\{(-\gamma_n)\|p_{n-1} - y\|^2\} + \inf_{y\in F}\{\gamma_n(1+\gamma_n)\|p_n - p_{n-1}\|^2\}$$

$$\le \inf_{y\in F}\{\|p_n - y\|^2\} + \gamma_n \inf_{y\in F}\{\|p_n - y\|^2\} - \gamma_n \inf_{y\in F}\{\|p_{n-1} - y\|^2\}$$

$$+\gamma_n (1+\gamma_n) \inf_{y \in F}\{\|p_n - y\|^2\}$$

$$\leq inf_{y \in F}\{\|p_n - y\|^2\} + \gamma_n[inf_{y \in F}\{\|p_n - y\|^2\} - inf_{y \in F}\{\|p_{n-1} - y\|^2\}]$$
$$+\gamma_n (1+\gamma_n) inf_{y \in F}\{\|p_n - y\|^2\} \qquad (25)$$

Now assume that $\varphi_n = \inf_{y \in F}\{\|p_n - y\|^2\}$ then

$$\varphi_{n+1} \leq \varphi_n + \gamma_n (\varphi_n - \varphi_{n-1}) + \delta_n \qquad (26)$$

Where $\delta_n = \gamma_n (1+\gamma_n) \|x_n - x_{n-1}\|^2$, then we observe that by (a) from (*)

$$\sum_{n=1}^{\infty} \delta_n = \sum_{n=1}^{\infty} \gamma_n (1+\gamma_n) \|p_n - p_{n-1}\|^2$$
$$\leq \sum_{n=1}^{\infty} \gamma_n (1+\gamma_n) (2K)^2$$
$$< \infty \qquad (27)$$

Now by using lemma 2.8 (b),

$\exists\, \varphi^* \in [0,\infty)$ such that $\lim_{n\to\infty} \varphi_n = \varphi^*$

$\Longrightarrow \lim_{n\to\infty} \inf_{y \in F}\{\|p_n - y\|^2\}$ exists. Therefore $\lim_{n\to\infty} \inf_{y \in F} \{\|p_n - y\|\}$ *also exists*.

Since T_1, T_2 and T_3 are all satisfy the condition *B*, so by theorem 3.2,

$\Rightarrow \lim_{n\to\infty} f\,(\lim_{y \in F}\{\|p_n - y\|\}) = 0$

and thus

$\lim_{n\to\infty} \lim_{y \in F}\{\|p_n - y\|\} = 0.$

So, from above we have a subsequence $\{p_{n_j}\}$ of $\{p_n\}$ and a sequence $\{p_j^*\} \subset F$ satisfies that

$$\left\|p_{n_j} - x_j^*\right\| < {}^{1}/_{2^j}$$

Now we will show that $\{p_j^*\}$ is a Cauchy sequence, for this let $\varepsilon > 0$, since $\lim_{n\to\infty} \lim_{y \in F}\{\|p_n - y\|\} = 0$, here $n \in N$ such that $\inf_{y \in F}\{\|p_n - y\|\} < {}^{\varepsilon}/_{6}, \forall\, m, n \geq N$, we have

$\|p_m - p_n\| \leq \|p_m - y\| + \|p_n - y\| \;\forall\, y \in F.$

Thus, we have

$$\|p_m - p_n\| \leq \lim_{y \in F}\{\|p_n - y\| + \|p_n - y\|\}$$
$$= \lim_{y \in F}\{\|p_m - y\|\} + \lim_{y \in F}\{\|p_n - y\|\}$$
$$< {}^{\varepsilon}/_{6} + {}^{\varepsilon}/_{6}$$
$$= {}^{\varepsilon}/_{3} \quad, \forall\, m, n > N.$$

Now, let $M = \max(N, j_0)$, then for all $j > k \geq M$, we have

$$\|p_j^* - p_k^*\| \leq \left\|p_j^* - p_{n_j}\right\| + \left\|p_{n_j} - p_{n_k}\right\| + \|p_{n_k} - p_k^*\|$$

$$< {}^{\varepsilon}/_{3} + {}^{\varepsilon}/_{3} + {}^{\varepsilon}/_{3}$$
$$= \varepsilon.$$

There $\{p_j^*\}$ is a Cauchy sequence and so $\exists$ a point $l \in H$, such that $\{p_j^*\}$ converges to l. Since F is closed & $l \in F$, then as outcome we see that $\{p_j^*\}$ converges to l. since $\lim_{n\to\infty}\|p_n - l\|$ exists by theorem 3.1, conclusion follows.

4. Conclusion

In this work, we introduce a new type iterative technique for finding fixed point and common fixed point.

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