

# Refinement of S.Bernstein Inequality

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**Abstract.** In the present paper we will discuss refinements of S.Bernstein's Inequality for the polynomials and will prove some results which will among other things also generalize.

**Keywords.** Polynomial, Derivative, Inequality

## Introduction

Suppose  $F(x)$  be a polynomial of degree  $m$  and  $F'(x)$  be its derivative. Concerning the estimate modulus of  $F'(x)$  on the unit circle  $|x|=1$  we know the inequality called as Bernstein's inequality

$$\max_{|x|=1} |F'(x)| \leq m \max_{|x|=1} |F(x)| \quad (1.1)$$

Concerning the estimate modulus of  $F(x)$  on a large circle  $|x|=R > 1$ , we get

$$\max_{|x|=R>1} |F(x)| \leq R^m \max_{|x|=1} |F(x)| \quad (1.2)$$

Inequality (1.1) is an consequence of S.Bernstein's theorem on he derivative of a trigonometric polynomials. Inequality (1.2) is a simple deduction consequence of maximum modulus principle.

For the both (1.1) and (1.2) holds for the polynomial  $F(x) = \beta x^m$ ,  $|\beta| \neq 0$ , that is, if and only if  $F(x)$  has all its zeros at the origin. It has been proved by Frappier, Ruscheweyh and Rahman that if  $F(x)$  is a polynomial of degree  $m$ , then

$$\max_{|x|=1} |F'(x)| \leq m \max_{1 \leq k \leq 2m} \left| F(e^{\frac{ik\pi}{m}}) \right| \quad (1.3)$$

Equation (1.3) clears represents a refinement of 91.1). since the maximum of  $|F(x)|$  on the  $|x|=1$  may be large than the maximum of  $F(x)$  taken over the  $2n^{\text{th}}$  roots of unity. Take an example  $F(x) = x^m + ib$ ,  $b > 0$ . As it has been proved by the A.Aziz interesting refinement of (1.3) and hence Bernstein's Inequality(1.1) as well.

**Theorem1.1:** If  $F(x)$  is a polynomial of degree  $m$ , then for every given real  $\beta$

$$\max_{|x|=1} |F'(x)| \leq \frac{m}{2} [M_{\beta} + M_{\beta+\pi}] \quad (1.4)$$

$$\text{Where } M_{\beta} = \max_{1 \leq k \leq m} \left| F(e^{\frac{i(\beta+2k\pi)}{m}}) \right| \quad (1.5)$$

$M_{\beta+\pi}$  is obtained from (1.5) by replacing  $\beta$  by  $\beta + \pi$ . The result is best possible and equality in (1.4) holds for  $F(x) = x^m + re^{i\beta}$ ,  $1 \leq r \leq 1$ .

**Theorem1.2** If  $F(x)$  is a polynomial of degree  $m$ , then for all real  $\beta$  and  $R > 1$ .

$$\max_{|x|=1} |F(Rx) - F(x)| \leq \left( \frac{R^n - 1}{2} \right) (M_{\beta} + M_{\beta+\pi}) \quad (1.6)$$

The result is best possible and equality in (1.6) holds for the polynomial  $F(x) = x^m + re^{i\beta}$ ,  $-1 \leq r \leq 1$ . If we restrict ourselves to the class of polynomials having no zero in  $|x| < 1$ , inequality (1.1) is sharpened. In fact P.Erdos conjectured and later P.D.Lax[5] verified that

$$\max_{|x|=1} |F'(x)| \leq \frac{m}{2} \max_{|x|=1} |F(x)| \quad (1.7)$$

**Theorem 1.3** If  $F(x)$  is a polynomial of degree  $m$  having no zero in  $|x| < 1$ , then for every real  $\beta$

$$\max_{|x|=1} |F'(x)| \leq \frac{m}{2} [M_\beta^2 + M_{\beta+\pi}^2]^{\frac{1}{2}} \quad (1.8)$$

The result is best possible and equality in (1.8) holds for  $F(x) = x^m + e^{i\beta}$ .

**Theorem 1.4** If  $F(x)$  is a polynomial of degree  $m$  having no zero in  $|x| < 1$ , then for every real  $\beta$  and  $R > 1$

$$\max_{|x|=1} |F(Rx) - F(x)| \leq \left( \frac{R^n - 1}{2} \right) (M_\beta^2 + M_{\beta+\pi}^2)^{\frac{1}{2}} \quad (1.9)$$

The result is sharp and equality in (1.9) holds for  $F(x) = x^m + e^{i\beta}$

Now we will prove the theorem one by one.

**Theorem A.** If  $F(x)$  is a polynomial of degree  $m$  having all its zeros in  $|x| \geq k \geq 1$ , then

$$\max_{|x|=1} |F'(x)|^2 \leq \frac{n^2}{2(1+k^2)} [M_\beta^2 + M_{\beta+2}^2] \quad (1.10)$$

Where  $M_\beta$  is defined by (1.5)

Taking  $k=1$ , Theorem A reduces to Theorem 1.3.

**Theorem B.** If  $F(x)$  is a polynomial of degree  $n$  having all its zeros in  $|x| \geq k \geq 1$ , then for all real  $\alpha$  and  $R > 1$ ,

$$|F(Rx) - F(x)| \leq \frac{R^n - 1}{\sqrt{2(1+k^2)}} [M_\beta^2 + M_{\beta+\pi}^2]^{\frac{1}{2}} \quad (1.11)$$

Where  $M_\beta + M_{\beta+\pi}$  are defined as in Theorem 1.1

**Corollary 1.** If  $F(x)$  is a polynomial of degree  $m$ , then for all real  $\beta$  and  $r \leq 1$ ,

$$\max_{|x|=1} |F(rz) - r^n F(x)| \leq \frac{1-r^n}{\sqrt{2(1+k^2)}} [M_\beta^2 + M_{\beta+\pi}^2]^{\frac{1}{2}} \quad (1.12)$$

**Theorem C.** If  $F(x)$  is a polynomial of degree  $m$  having all its zeros in  $|x| < k, k \leq 1$  then

$$\max_{|x|=1} |F'(x)| \leq \frac{n}{\sqrt{2(1+k^2)}} [M_\beta^2 + M_{\beta+\pi}^2]^{\frac{1}{2}} \quad (1.13)$$

**Theorem D.** If  $F(x)$  is a polynomial of degree  $m$  having all its zeros on  $|x| \leq k, k \leq 1$ , then for all real  $\alpha$  and  $R > 1$ ,

$$|F(Rx) - F(x)| \leq \frac{R^n - 1}{\sqrt{2(1+k^{2n})}} [M_\beta + M_{\beta+\pi}]^{\frac{1}{2}} \quad (1.14)$$

**Theorem E.** If  $F(x)$  is self inverse polynomial of degree  $m$ , then

$$\max_{|x|=1} |F'(x)| \leq \frac{n}{2} \sqrt{M_\beta^2 + M_{\beta+\pi}^2}, \quad (1.15)$$

where  $M_\beta$  is defined by (1.5)

**Theorem F:** If  $F(x)$  is a polynomial of degree  $m$  such that  $F(1)=0$ , then for  $0 \leq w \leq m$ ,

$$\left| F\left(1 - \frac{w}{m}\right) \right| \leq \frac{1}{2} \left[ 1 - \left(1 - \frac{w}{m}\right)^m \right] (M_0 + M_\pi)$$

The result is best possible if  $F(x) = x^m - 1$

**Proof of theorem: for the proof of these theorems we need the following lemmas**

**Leema1:** If  $F(x)$  is a polynomial of degree  $m$ , then for  $|x| = 1$  and for every real  $\beta$ ,

$$|F'(x)|^2 + |mF(x) - xF'(x)|^2 \leq \frac{m^2}{2} [M_\beta^2 + M_{\beta+\pi}^2] \quad (1.16)$$

Where  $[M_\beta^2 + M_{\beta+\pi}^2]$  are defined as in Theorem 1.

**Leema2:** If  $F(x)$  is a polynomial of degree  $m$  having all its zeros in  $|x| \geq k \geq 1$ , then

$$k^s |F(e^{i\theta})| \leq |Q^s(e^{i\theta})|, 0 \leq \theta \leq 2\pi, \quad (1.17)$$

Where  $Q(x) = \overline{znF\left(\frac{1}{x}\right)}$

**Leema3:** If  $F(x)$  is a polynomial of degree  $m$  having all its zeros in  $|x| < k, k \leq 1$ , then

$$km \max_{|x|=1} |F'(x)| \leq \max_{|x|=1} |Q'(x)| \quad (1.18)$$

Where  $Q(x)$  is as mentioned in Leema 2.

**Proof of theorem A:** Let  $Q(x) = \overline{xnF\left(\frac{1}{x}\right)}$ . Then

$$|Q'(x)| = |mF(x) - xF'(x)|, \text{ for } |x| = 1$$

By using (1.10), we will get

$$|F'(x)| + |Q'(x)| \leq \frac{m^2}{2} [M_\beta^2 + M_{\beta+\pi}^2] \quad (1.19)$$

From equation (1.17) with  $s=1$ , we have

$$k|F'(x)| \leq |Q'(x)|, \text{ for } |x| = 1$$

Hence

$$\begin{aligned} (1+k^2)|F'(x)|^2 &= |F'(x)|^2 + k^2|F'(x)|^2 \\ &\leq |F'(x)|^2 + |Q'(x)|^2 \\ &\leq \frac{m^2}{2} [M_\beta^2 + M_{\beta+\pi}^2] \end{aligned}$$

This gives

$$|F'(x)|^2 \leq \frac{m^2}{2(1+k^2)} [M_\beta^2 + M_{\beta+\pi}^2] \quad (1.20)$$

**Hence proved Theorem A.**

**Proof of Theorem B:** We have  $\forall t \geq 1$  and  $0 \leq \theta \leq 2\pi$

$$|F'(te^{i\theta})| \leq t^{n-1} \max_{|x|=1} |F'(x)| \quad (1.21)$$

Now applying Theorem 1 to the polynomial  $F(x)$  which is of degree  $m-1$ , we will get

$$|F'(tei\theta)| \leq t^{m-1} \frac{m}{\sqrt{2(1+k^2)}} [M_\beta^2 + M_{\beta+\pi}^2]^{\frac{1}{2}}$$

Hence for each  $\theta, 0 \leq \theta \leq 2\pi$  and  $R > 1$ , we have

$$\begin{aligned}
|F(\mathbf{R}e^{i\theta}) - F(e^{i\theta})| &= \left| \int_1^{\mathbf{R}} e^{i\theta} F'(te^{i\theta}) dt \right| \\
&\leq \int_1^{\mathbf{R}} |F'(te^{i\theta})| dt \\
&\leq \frac{m}{\sqrt{2(1+k^2)}} [M_\beta^2 + M_{\beta+\pi}^2]^{\frac{1}{2}} \int_1^{\mathbf{R}} t^{m-1} dt \\
&= \frac{\mathbf{R}^m - 1}{\sqrt{2(1+k)^2}} [M_\beta^2 + M_{\beta+\pi}^2]
\end{aligned}$$

This gives

$$|F(\mathbf{R}x) - F(x)| \leq \frac{\mathbf{R}^m - 1}{\sqrt{2(1+k)^2}} [M_\beta^2 + M_{\beta+\pi}^2]^{\frac{1}{2}}$$

Hence we get the required result.

**Proof of Theorem C.** We have from Lemma 1

$$\begin{aligned}
|F'(x)|^2 + |mF(x) - xF'(x)| &\leq \frac{m^2}{2} [M_\beta^2 + M_{\beta+\pi}^2] \tag{1.22} \\
(1+k^{2m}) \max_{|x|=1} |F'(x)|^2 &= |F'(x)|^2 + k^{2m} |F'(x)|^2 \\
&= |F'(x)|^2 + |k^m F'(x)|^2 \\
(1+k^{2m}) \max_{|x|=1} |F'(x)|^2 &\leq \max_{|x|=1} [|F'(x)|^2 + |Q'(x)|^2] \\
&= \max_{|x|=1} [|F'(x)|^2 + |mF(x) - xF'(x)|^2] \\
&\leq \frac{m^2}{2} [M_\beta^2 + M_{\beta+\pi}^2] \\
\max_{|x|=1} |F'(x)| &\leq \frac{m}{\sqrt{2(1+k^{2m})}} [M_\beta^2 + M_{\beta+\pi}^2]^{\frac{1}{2}}
\end{aligned}$$

Hence completes the Proof of the theorem.

**Proof of Theorem D.**

We have  $\forall t \geq 1$  and  $0 \leq \theta \leq 2\pi$

$$|F'(te^{i\theta})| \leq t^{n-1} \max_{|x|=1} |F'(x)|$$

Now applying Theorem 1 to the polynomial  $F(x)$  which is of degree  $m-1$ , we will get

$$|F'(tei\theta)| \leq t^{m-1} \frac{m}{\sqrt{2(1+k^2)}} [M_\beta^2 + M_{\beta+\pi}^2]^{\frac{1}{2}}$$

Hence for each  $\theta, 0 \leq \theta \leq 2\pi$  and  $\mathbf{R} > 1$ , we have

$$\begin{aligned}
|F(\mathbf{R}e^{i\theta}) - F(e^{i\theta})| &= \left| \int_1^{\mathbf{R}} e^{i\theta} F'(te^{i\theta}) dt \right| \\
&\leq \int_1^{\mathbf{R}} |F'(te^{i\theta})| dt
\end{aligned}$$

$$\begin{aligned} &\leq \frac{m}{\sqrt{2(1+k^2)}} \left[ M_\beta^2 + M_{\beta+\pi}^2 \right]^{\frac{1}{2}} \int_1^R t^{m-1} dt \\ &= \frac{R^m - 1}{\sqrt{2(1+k)^2}} \left[ M_\beta^2 + M_{\beta+\pi}^2 \right] \end{aligned}$$

This gives

$$|F(Rx) - F(x)| \leq \frac{R^m - 1}{\sqrt{2(1+k)^2}} \left[ M_\beta^2 + M_{\beta+\pi}^2 \right]^{\frac{1}{2}}$$

Hence we get the required result.

**Proof of Theorem E.** Now  $F(x) = xmF\left(\frac{1}{x}\right)$

We have

$$F'(x) = mx^{m-1}F\left(\frac{1}{x}\right) - x^{m-2}F\left(\frac{1}{x}\right)$$

$$xF'(x) = mx^mF\left(\frac{1}{x}\right) - x^{m-1}F\left(\frac{1}{x}\right)$$

$$xF'(x) = mF(x) - x^{m-1}F\left(\frac{1}{x}\right)$$

$$|mF(x) - xF'(x)| = |F'(x)|, \text{ for } |x| = 1$$

By using lemma 1 we have

$$2|F'(x)|^2 = |F'(x)|^2 + |mF(x) - xF'(x)|^2$$

$$|F'(x)|^2 \leq \frac{m^2}{4} \left[ M_\beta^2 + M_{\beta+\pi}^2 \right]$$

$$|F'(x)| \leq \frac{m}{2} \left[ M_\beta^2 + M_{\beta+\pi}^2 \right]^{\frac{1}{2}},$$

Hence proved.

**Proof of Theorem F.** If  $Q(x) = x^m F\left(\frac{1}{x}\right)$

Then  $|Q(x)| = |F(x)|$  for  $|x| = 1$

and by the use of hypothesis we have

$$Q(1) = \overline{F(1)} = 0 \text{ by using } Q(x) \text{ with } \beta = 0, \text{ we get for } R > 1$$

$$|Q(R)| \leq \frac{R^m - 1}{2} [M_0 + M_\pi] \text{ for } R > 1$$

$$\left| F\left(\frac{1}{R}\right) \right| \leq \frac{1}{2} (1 - R^{-m}) (M_0 + M_\pi)$$

For  $0 \leq w \leq m$ , then  $\left(1 - \frac{w}{m}\right)^{-1} > 1$  and in particular, we have

$$\left| F\left(1 - \frac{w}{m}\right) \right| \leq \frac{1}{2} \left[ 1 - \left(1 - \frac{w}{m}\right)^m \right] (M_0 + M_\pi)$$

Hence proved.

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