

A generalized starlike functions with respect to symmetric points

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Abstract: The purpose of the present paper is to estimate the Certain Coefficient for generalized Starlike functions with respect to symmetric points defined on the open unit disk for which $R_{\lambda, \delta}^k(\phi)$ of normalized analytic functions $f(z)$ lies in a region with respect to 1 and symmetric with respect to the real axis.

Keywords: Analytic function, Univalent function, Starlike function, Convex function, Subordination, Hadamard product, Linear operator, Fekete-szegő Inequality.

INTRODUCTION

Let A denote the class of all analytic function $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the condition $f(0) = 0, f'(0) = 1$. We also denote by S the subclass of A consisting of all functions which are univalent in U . For functions $f(z)$ and $g(z)$ analytic in U , we say that the functions $f(z)$ is said to subordinate to $g(z)$ if there exist a schwarz function $\omega(z)$, analytic in U with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in U),$$

such that

$$f(z) = g(\omega(z)) \quad (z \in U).$$

We denote this subordination by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in U).$$

In particular, if the function $g(z)$ is univalent in U , the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Let $\phi(z)$ be an analytic function in U with $\phi(0) = 1, \phi'(0) > 0$ and $Re\{\phi(z)\} > 0, z \in U$ which map the open unit disk U onto a region starlike with respect to 1 and is symmetric with respect to the real axis. Then by $S^*(\phi)$ and $C(\phi)$, respectively, we denote the subclasses of the normalized analytic function class A , which satisfy the following subordination relations:

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \quad z \in U$$

and

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), z \in U.$$

These function were introduced and studied by Ma and Minda[7]. In particular case, when

$$\phi(z) = \frac{1+(1-2\alpha)z}{1-z}, z \in U, 0 \leq \alpha < 1,$$

these function reduce respectively to the well-known classes $S^*(\alpha)$, ($0 \leq \alpha < 1$) of starlike functions of order α in U and $C(\alpha)$, ($0 \leq \alpha < 1$) of convex functions of order α in U . Ma and Minda [7], the Fekete-Szegő inequality for the functions in the class $C(\phi)$ was derived and in view of the Alexander result relating the function classes $S^*(\phi)$ and $C(\phi)$.

For a brief history of the Fekete-Szegő problems for the starlike, convex and other various subclasses of the normalized analytic function in A , we refer the reader to the work done by Srivatsava et al [17]. of course the main result shall refer back to Fekete and szegő [2] in the year 1933. After 30 years or so, Keogh and Merkes [4] solved the problem for certain subclasses of univalent functions.

Koepf [5], gave excellent results for the class of close-to-convex functions. Recently Shanmugam et al [14] have studied the Fekete-Szegő problem for subclasses of starlike functions with respect to symmetric points. Motivated essentially by the aforementioned works, we prove the Fekete-Szegő inequality for a more general class of normalized analytic functions.

For $\lambda, \delta \in N$, $k \in N_0$ the authors Darus [1] introduced the operator $D_{\lambda, \delta}^k$ defined by

$$D_{\lambda, \delta}^k f(z) = z + \sum_{n=2}^{\infty} [1+(n-1)\lambda]^k C(\delta, n) a_n z^n \quad (1.2)$$

In the present paper, we obtain the Fekete-szegő inequality for the function $f \in A$ in the class $R_{\lambda, \delta}^k(\phi)$ defined as follows:

Definition 1 Let $D_{\lambda, \delta}^k : A \rightarrow A$ is a linear operator and $D_{\lambda, \delta}^k$ is analytic in $f(U)$.

Let

$$D_{\lambda, \delta}^k f(z) = z + \sum_{n=2}^{\infty} [1+(n-1)\lambda]^k C(\delta, n) a_n z^n$$

where

$$C(\delta, n) = \frac{\Gamma(n+\delta)}{\Gamma(n) \Gamma(\delta+1)}$$

when $\lambda = 1$, $\delta = 0$ we get the *sălăgean* differential operator, $k = 0$ or $\lambda = 0$ gives Ruscheweyh operator, $\delta = 0$ gives Al-oboudi differential operator of order k

$$D_{1,0}^0 f(z) = f(z), D_{1,0}^1 f(z) = zf'(z)$$

Definition 2 Let $\phi(z)$ be a univalent starlike function with respect to 1 which map the unit disk U onto a region in the right half plane which is symmetric with respect to the real axis $\phi(0) = 1$ and $\phi'(0) > 1$. A function $f \in A$ is in the class $R_{\lambda, \delta}^k(\phi)$ if

$$\frac{(s-t)z \left[D_{\lambda, \delta}^k f(z) \right]}{D_{\lambda, \delta}^k [f(sz)] - D_{\lambda, \delta}^k [f(tz)]} \prec \phi(z), \quad (\lambda, \delta \in \mathbb{N}, k \in \mathbb{N}_0).$$

In order to prove our main results, we need the following lemma.

Lemma 1.1 [7] *If $p_1(z) = 1 + c_1z + c_2z^2 + \dots$ is an analytic function with positive real part in \mathbb{U} , then*

$$|c_2 - \nu c_1^2| \leq \begin{cases} -4\nu + 2, & \text{if } \nu \leq 0 \\ 2, & 0 \leq \nu \leq 1 \\ 4\nu - 2, & \nu \geq 1. \end{cases}$$

when $\nu < 0$ or $\nu > 1$, the equality holds true if and only if $p_1(z) = \frac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$, then the equality holds true if and only if $p_1(z) = \frac{1+z^2}{1-z^2}$ or one of its rotation.

If $\nu = 0$, then the equality holds true if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\gamma \right) \frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\gamma \right) \frac{1-z}{1+z} \quad (0 \leq \gamma \leq 1),$$

or one of its rotations. If $\nu = 1$, the equality holds if and only if p_1 is the reciprocal of one of the functions such that the equality holds in the case of $\nu = 0$. Also the above upper bound is sharp, it can be improved as follows when $0 < \nu < 1$:

$$|c_2 - \nu c_1^2| + \nu |c_1^2| \leq 2, \quad 0 < \nu \leq \frac{1}{2} \quad \text{and} \quad |c_2 - \nu c_1^2| + (1-\nu) |c_1^2| \leq 2, \quad \frac{1}{2} < \nu \leq 1$$

We also need the following result in our investigation.

Lemma 1.2 [12] *If $p_1(z) = 1 + c_1z + c_2z^2 + \dots$ is a function with positive real part in \mathbb{U} , then $|c_2 - \nu c_1^2| \leq 2 \max(1, |2\nu - 1|)$.*

The result is sharp for the functions $p_1(z)$ given by $p_1(z) = \frac{1+z^2}{1-z^2}$ and $p_1(z) = \frac{1+z}{1-z}$.

FEKETE-SZEGÖ PROBLEM FOR THE FUNCTION OF THE CLASS $R_{\lambda, \delta}^k(\phi)$

By making use of Lemma 1.2, we prove the Fekete-szegö Problem for the class $R_{\lambda, \delta}^k(\phi)$

Theorem 2.1 *Let $\phi(z) = 1 + B_1z + B_2z^2 + \dots$. If $f(z)$ given by (1.1) belongs to the class $R_{\lambda, \delta}^k(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \Lambda, & \mu \leq \sigma_1 \\ \eta, & \sigma_1 \leq \mu \leq \sigma_2 \\ -\Lambda, & \mu \geq \sigma_2. \end{cases}$$

where

$$\sigma_1 = \frac{(\delta+1)(s+t-2)(1+\lambda)^{2k}}{2(\delta+2)(1+2\lambda)^k[(s^2+st+t^2)-3]} \left[\frac{2(B_2 - B_1)(s+t-2) - B_1^2(s+t)}{B_1^2} \right],$$

$$\sigma_2 = \frac{(\delta+1)(s+t-2)(1+\lambda)^{2k}}{2(\delta+2)(1+2\lambda)^k[(s^2+st+t^2)-3]} \left[\frac{2B_2(s+t-2) - (s+t)B_1^2}{B_1^2} \right],$$

$$\sigma_3 = \frac{(\delta+1)(s+t-2)(1+\lambda)^{2k}}{2(\delta+2)(1+2\lambda)^k[(s^2+st+t^2)-3]} \left[\frac{(2B_2 - B_1)(s+t-2) - B_1^2(s+t)}{B_1^2} \right],$$

$$\Lambda = \frac{4(\delta+1)(\delta+2)(1+2\lambda)^k[3 - (s^2 + st + t^2)]}{B_1^2}$$

$$\times \left[(B_2 - B_1^2) - \frac{B_1^2}{2} \left(\frac{s+t}{s+t-2} + \frac{2\mu(1+2\lambda)^k(\delta+2)[(s^2+st+t^2)-3]}{(\delta+1)(1+\lambda)^{2k}(2-s-t)^2} \right) \right],$$

$$\eta = \frac{2B_1}{(1+2\lambda)^k(\delta+1)(\delta+2)(3 - [s^2 + st + t^2])}.$$

Further,

If $\sigma_1 \leq \mu \leq \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{(\delta+1)(s+t-2)(1+\lambda)^{2k}}{(\delta+2)(1+2\lambda)^k[(s^2+st+t^2)-3]B_1^2} \{(B_1 - B_2)(s+t-2) + \sigma_4 B_1^2\} |a_2|^2 \leq \eta.$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{(\delta+1)(s+t-2)(1+\lambda)^{2k}}{(\delta+2)(1+2\lambda)^k[(s^2+st+t^2)-3]B_1^2} \{B_2(s+t-2) - \sigma_4 B_1^2\} |a_2|^2 \leq \eta.$$

Where

$$\sigma_4 = \left[\frac{(s+t)(s+t-2)(\delta+1)(1+\lambda)^{2k} + \mu(\delta+2)(1+2\lambda)^k[(s^2+st+t^2)-3]}{(\delta+1)(s+t-2)(1+\lambda)^{2k}} \right].$$

The result is sharp.

Proof. If $f \in R_{\lambda, \delta}^k(\phi)$, then there exists a Schwarz function $w(z)$, analytic in U with $w(0) = 0$ and $|w(z)| < 1$ in U such that

$$\frac{(s-t)z \left[D_{\lambda, \delta}^k f(z) \right]}{D_{\lambda, \delta}^k [f(sz)] - D_{\lambda, \delta}^k [f(tz)]} = \phi(w(z))$$

Define a function $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)}.$$

since $w(z)$ is a Schwarz function, we see that $\operatorname{Re}\{p_1(z)\} > 0$ and $p_1(0) = 0$. Define a function $p(z)$ by

$$p(z) = \frac{(s-t)z \left[D_{\lambda, \delta}^k f(z) \right]}{D_{\lambda, \delta}^k [f(sz)] - D_{\lambda, \delta}^k [f(tz)]} = \phi(w(z)) = 1 + b_1 z + b_2 z^2 + \cdots \quad (2.1)$$

From (2.1), we obtain

$$a_2 = \frac{b_1}{(1+\lambda)^k (\delta+1)(2-s-t)} \quad (2.2)$$

and

$$a_3 = 2 \frac{b_2 - (s+t)(s+t-2)(1+\lambda)^{2k} (\delta+1)^2 a_2^2}{[3 - (s^2 + st + t^2)](1+2\lambda)^k (\delta+1)(\delta+2)}$$

since

$$p_1(z) = \frac{1 + \phi^{-1}(p(z))}{1 - \phi^{-1}(p(z))}$$

then

$$p(z) = \phi \left[\frac{p_1(z) - 1}{p_1(z) + 1} \right],$$

and

$$\begin{aligned} 1 + b_1 z + b_2 z^2 + \cdots &= \phi \left(\frac{c_1 z + c_2 z^2 + \cdots}{2 + c_1 z + c_2 z^2 + \cdots} \right) \\ &= \phi \left[\frac{1}{2} c_1 z + \frac{1}{2} (c_2 - \frac{1}{2} c_1^2) z^2 + \cdots \right] \quad (2.3) \end{aligned}$$

Equating the coefficients of z and z^2 , we obtain

$$b_1 = \frac{1}{2} B_1 c_1 \quad \text{and} \quad b_2 = \frac{1}{2} B_1 (c_2 - \frac{1}{2} c_1^2) + \frac{1}{4} B_2 c_1^2 \quad (2.4)$$

From (2.2) and (2.4), we get

$$a_2 = \frac{B_1 c_1}{2(1+\lambda)^k (\delta+1)(2-s-t)}$$

$$\text{and } a_3 = \frac{B_1}{(1+2\lambda)^k (\delta+1)(\delta+2)[3 - (s^2 + st + t^2)]} \left\{ c_2 - c_1^2 \left(1 - \frac{1}{2} \frac{B_2}{B_1} + \frac{1}{2} B_1 \left(\frac{s+t}{s+t-2} \right) \right) \right\}$$

Therefore we have

$$|a_3 - \mu a_2^2| = \frac{B_1}{[3 - (s^2 + st + t^2)](1 + 2\lambda)^k (\delta + 1)(\delta + 2)} \left\{ c_2 - c_1^2 \left(1 - \frac{1}{2} \frac{B_2}{B_1} + \frac{1}{2} B_1 \left(\frac{s+t}{s+t-2} \right) \right) \right\} \\ - \frac{\mu B_1^2 c_1^2}{(\delta + 1)^2 (1 + \lambda)^{2k} (2 - s - t)} \\ = \frac{B_1}{[3 - (s^2 + st + t^2)](1 + 2\lambda)^k (\delta + 1)(\delta + 2)} \{c_2 - \nu c_1^2\}$$

where

$$\nu = \left[1 - \frac{1}{2} \frac{B_2}{B_1} + \frac{B_1}{2} \left(\frac{s+t}{s+t-2} \right) - \frac{\mu B_1 (\delta + 2)(1 + 2\lambda)^k [3 - (s^2 + st + t^2)]}{(\delta + 1)(1 + \lambda)^{2k} (2 - s - t)^2} \right]$$

If $\mu \leq \sigma_1$, then by Lemma 1.1 and Lemma 1.2, we obtain

$$|a_3 - \mu a_2^2| \leq \frac{4(\delta + 1)(\delta + 2)(1 + 2\lambda)^k [3 - (s^2 + st + t^2)]}{B_1^2} \\ \left[(B_2 - B_1) - \frac{B_1^2}{2} \left(\frac{s+t}{s+t-2} + \frac{2\mu(\delta + 2)(1 + 2\lambda)^k [(s^2 + st + t^2) - 3]}{(\delta + 1)(1 + \lambda)^{2k} (2 - s - t)^2} \right) \right]$$

which is the first part of Theorem 1.1.

Similarly, if $\mu \geq \sigma_2$, then by Lemma 1.1 and Lemma 1.2, we obtain

$$|a_3 - \mu a_2^2| \leq \frac{4(\delta + 1)(\delta + 2)(1 + 2\lambda)^k [3 - (s^2 + st + t^2)]}{B_1^2} \\ \left[(B_1 - B_2) + \frac{B_1^2}{2} \left(\frac{s+t}{s+t-2} + \frac{2\mu(\delta + 2)(1 + 2\lambda)^k [(s^2 + st + t^2) - 3]}{(\delta + 1)(1 + \lambda)^{2k} (2 - s - t)^2} \right) \right]$$

If $\sigma_1 \leq \mu \leq \sigma_2$, we see that

$$|a_3 - \mu a_2^2| = \frac{2B_1}{2(1 + 2\lambda)^k (\delta + 1)(\delta + 2)(3 - [s^2 + st + t^2])} \{c_2 - \nu c_1^2\} \\ \leq \frac{2B_1}{\delta + 1(\delta + 2)(1 + 2\lambda)^k [3 - (s^2 + st + t^2)]}$$

Further, If $\sigma_1 \leq \mu \leq \sigma_3$, then

$$|a_3 - \mu a_2^2| + (\mu - \sigma_1) |a_2^2| \leq \frac{2B_1}{\delta + 1(\delta + 2)(1 + 2\lambda)^k [3 - (s^2 + st + t^2)]}$$

Finally, we see that

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_3 - \mu a_2^2| + (\sigma_2 - \mu) |a_2^2| \leq \frac{2B_1}{\delta + 1(\delta + 2)(1 + 2\lambda)^k [3 - (s^2 + st + t^2)]}$$

To show that the bounds are sharp, we define functions k_n^ϕ ($n = 2, 3, \dots$) by

$$\frac{z(D_{\lambda, \delta}^k k_n^\phi(z))'}{D_{\lambda, \delta}^k k_n^\phi(z)} = \phi(z^{n-1}), k_n^\phi(0) = 0 = (k_n^\phi(0))' - 1,$$

and the function F_γ and G_γ ($0 \leq \gamma \leq 1$) by

$$\frac{z(D_{\lambda, \delta}^k F_\gamma(z))'}{D_{\lambda, \delta}^k F_\gamma(z)} = \phi\left(\frac{z(z + \gamma)}{1 + \gamma z}\right), F_\gamma(0) = 0 = (F_\gamma(0))' - 1$$

and

$$\frac{z(D_{\lambda, \delta}^k G_\gamma(z))'}{D_{\lambda, \delta}^k G_\gamma(z)} = \phi\left(-\frac{z(z + \gamma)}{1 + \gamma z}\right), G_\gamma(0) = 0 = (G_\gamma(0))' - 1$$

Clearly the functions k_n^ϕ, F_γ and $G_\gamma \in R_{\lambda, \delta}^k(\phi)$. We also write $K^\phi = K_2^\phi$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality in Theorem 2.1 holds true if and only if f is K^ϕ or one of its rotations.

When $\sigma_1 < \mu < \sigma_2$, then the equality holds true if and only if f is K_3^ϕ or one of its rotations.

If $\mu = \sigma_1$, then the equality holds true if and only if f is F_γ or one of its rotations. If $\mu = \sigma_2$, then the equality holds true if and only if f is G_γ or one of its rotations.

By making use of Lemma 1.2, we can easily obtain the following theorem.

Theorem 2.2 Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \dots$, where the coefficients B_n are real with $B_1 > 0$ and $B_2 \geq 0$. If $f(z)$ given by (1.1) belongs to $R_{\lambda, \delta}^k(\phi)$, then

$$|a_3 - \mu a_2^2| \leq \left(\frac{4B_1}{(1 + 2\lambda)^k (\delta + 1)(\delta + 2)[3 - (s^2 + st + t^2)]} \right) \times \max \left\{ 1, \left| 1 - \frac{2B_2}{B_1} + B_1 \left(\frac{s+t}{s+t-2} - \frac{2\mu(\delta+2)(1+2\lambda)^k [3 - (s^2 + st + t^2)]}{(\delta+1)(2-s-t)^2 (1+\lambda)^{2k}} \right) \right| \right\}, \mu \in \mathbb{C}$$

The result is Sharp.

Remark 1 The coefficient bounds for $|a_2|$ and $|a_3|$ are special cases of those asserted by Theorem 2.1

Remark 2 In its special case when $\lambda = 1$, $\delta = 0$ and $k = 0$, we arrive at a known result due to Ma and Minda [7]

Remark 3 In its special case when $\lambda = 1$, $\delta = 0$ and $k = 0, s = 1, t = -1$, we arrive at a known result due to T.N Shanmugam et al [14]

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