COMMON FIXEDPOINT THEOREMS USINGFOUR-STEP ITERATION SCHEMES IN BANACH SPACES FOR G-NONEEXPANSIVE MAPPINGS WITH A GRAPH

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Abstract

The purpose of this article to establish weak and strong convergence theorem of a new modified four-step iteration in a uniformly convex Banach space for four G-nonexpansive mappings.

Keywords Directed graph, Uniformlyconvex Banach space, G-nonexpansive mappings, Three step Noor iteration, SP iteration.

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1. Introduction

Fixed point theory is the huge active area of research due to its many fruitful applications in various fields. It addresses the results which state that, under given condition a map on a set to itself admits a fixed point. In 1922 Banach [4]provedthe existence of a unique fixed point for contraction mappings in a complete metric space. Due to its applications in many branches of mathematics and other related fields Banach contraction principle has been generalized in many directions.RecentlyJachymski generalized its by the combination of the concepts in fixed point theory and graph theory,In 2008 Jachymski [8] generalized the Banach contraction principle in a complete metric space endowed with a directed graph. Applying this concept he proved the Kelisky-Rivlintheorem [10]. In 2012, Aleomarninejad et al. [1] studied some iterative scheme for G-contraction and G-nonexpansive multivalued mappings in a Banach space with a graph. In 2015, Alfuraidan and Khasmi [2] pressentedthe concept of G- monotone nonexpansive multivalued mappings defined on a hyperbolic metric space with a graph. Alfuraidan [3] defined the existence of fixed point of monotone nonexpansive mappings on a Banach space endowed with a directed graph. Tiammee et al. [18] gives Browder's convergence theorem for G-nonexpansive mappings in a Hilbert space with a directed graph. They also proved strong convergence of the Halpern iteration for a G-nonexpansive mapping.

Due to Tripak [19]in 2016 the Ishikawa iterative scheme (usually called two step iteration) is given by:

$$b_n = (1-\beta_n)x_n + \beta_n S_1 a_n$$

$$a_{n+1} = (1-\alpha_n)a_n + \alpha_n S_2 b_n, n \geq 0 \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in [0,1]. The Ishikawa iteration method (1.1) is used to estimate common fixed point of two G-nonexpansive mappings in a Banach space endowed with a graph.

In 2000, Noor [11]gives the convergence criteria of three-stepiteration method for solving general variational inequalities and elated problems. Glowinski and Le Tallec [6] studied three step iteration method for find the approximate solutions of the elastoviscoplasticity problem, eigenvalue computation and liquid crystal theory. It has been shown by Glowinski et al. that the three-step iterative method gives better numerical results than the two-step and one-step approximate iterations. In 1998, Haubruuge et al. [7] investigate the convergence analysis of three-step methods of Glowinski and Le Tallec [6] and applied these methods to obtain new splitting-type algorithms for variation inequalities, minimization of a sum of convex functions and separable convex programming. They also showed that three-step iteration lead to highly parallelized algorithms under certain conditions.Thusit can be say that the three-step iteration method is a crucial part in solving various problems which arise in pure and applied sciences.

2. Preliminaries and Definitions

The three step Noor iteration method applying to approximate common fixed point of three G-nonexpansive mappings is given by:

$c_n = (1-\gamma_n)a_n + \gamma_n S_3 a_n,$

$b_n = (1-\beta_n)a_n + \beta_n S_2 c_n ,$

$c_{n+1} = (1-\alpha_n)a_n + \alpha_n S_1 b_n,\ n \geq 0,$ (2.1)

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in [0,1].

In 2017, Sridarat et al. [15] introduce an iterative method which is called the SP iteration for three G-nonexpansive mappings is given by:

$c_n = (1-\gamma_n)a_n + \gamma_n S_3 a_n ,$

$b_n = (1-\beta_n)c_n + \beta_n S_2 c_n ,$

$a_{n+1} = (1-\alpha_n)b_n + \alpha_n S_1 b_n,\ n \geq 0,$ (2.2)

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in [0,1]. They proved the strong and weak convergence for a uniformly convex Banach space endowed with a graph.

Recently Damrongask et al. [13] introduce a new modified three-step iteration method for three G-nonexpansive mappings given by:

$c_n = (1-\gamma_n)a_n + \gamma_n S_3 a_n ,$

$b_n = (1-\beta_n)a_n + \beta_n S_2 c_n ,$

$a_{n+1} = (1-\alpha_n)a_n + \alpha_n S_1 b_n,\ n \geq 0,$ (2.3)

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in [0,1]. They proved the strong and weak convergence for a uniformly convex Banach space endowed with a graph.

Motivated by the recent works, we introduce and study a new modified four-step iteration method for four G-nonexpansive mappings, where the sequence $\{a_n\}$ is generated by $a_0 \in$ C and:

$d_n = (1-\gamma_n)a_n + \gamma_n S_4 a_n ,$

$c_n = (1-\gamma_n)d_n + \gamma_n S_3 d_n ,$

$b_n = (1-\beta_n)c_n + \beta_n S_2 c_n ,$

$a_{n+1} = (1-\alpha_n)S_2 b_n + \alpha_n S_1 b_n,\ n \geq 0,$ (2.4)

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are realsequences in [0,1] and C is a convex subset of a normed space.

The main objective of this article to construct an iteration method for approximating common fixed point for four G-nonexpansive mappings and to study some weak and strong convergence theorems for such mappings in a uniformly convex Banach space endowed with a graph.

Let C be a nonempty subset of a real Banach space X and Δ denote the diagonal of the Cartesian product CxC, $i.e., \Delta = \{(a,a) : a \in C\}$. Consider a directed graph in such a way that the set V(G) of its vertices coincides with C, and the set E(G) of its edges contains all its self-loops, i.e., E(G)⊇Δ. We assume that G has no parallel edges. We can define the graph G with the pair (V(G), E(G)). We denote the conversion of G by G^{-1}, i.e., the graph obtained from G by reversing the direction of edges.Thus we have:

$$E(G^{-1}) = \{(a,b) \in X \times X : (b,a) \in E(G)\}.$$

We recall few basic concepts concerning the connectivity of graphs. All of them can be -found, e.g., in [9]. If a and b are vertices in graph G, then a path from a to$b$in G of length N ($N \in \mathbb{N} \cup \{0\}$) is a sequence $\{a_i\}_{i=0}^{N}$ of $N+1$ vertices such that $a_0 = a, a_N = b$ and (a_i, a_{i-1})∈E(G) for $i = 0,1, \dots N-1$.A directed graph G is said to be connected if there is path between any two vertices. A directed graph G is said to be transitive, if for any a,b,c∈ V(G) such that (a, b) and $(b, c)$are in E(G), that implies (a, c) in E(G).

Let A be a nonempty subset of V(G)and $a_0 \in$ V(G). We say that A is dominated by a_0 if (a_0, a) in E(G) for all $a \in$ A. If for each a∈ A,(a_0, a)∈ E(G) then A dominates a_0.

We say that a mapping S: C→C is said to be G- contraction if S satisfies the following conditions:

(i) S preserves edge of G (or S is edge preserving) i.e.,

(a, b)∈ E(G) ⇒(Sa, Sb) ∈ E(G),

(ii) S decrease weights of edges of G in the following conditions: there exist λ∈ (0,1) such that:

$(a,b) \in$ E(G) $\Rightarrow \|Sa - Sb\| \leq \lambda\|a - b\|$.

A mapping S: C→C is said to be G-nonexpansive(see [2], Definition 2.3(iii)) if S satisfies the flowing conditions:

(i) S preserves edge of G, i.e.,

$$(a,b) \in \text{E(G)} \Rightarrow \|Sa - Sb\| \in E(G)$$

(ii) S non-increases its weights of edges of G, i.e.,

$(a,b) \in$ E(G) $\Rightarrow \|Sa - Sb\| \leq \|a - b\|$.

In this paper, we denote →and ⇀ the strong convergence and weak convergence respectively.

If a mapping S: C → C defined in such a way that for any sequence $\{a_n\}$ in C such that$(a_n, a_{n+1}) \in$ E(G),$a_n \rightharpoonup a$ and S$a_n \to 0$ implies Sa =0, then S is G-demiclosed at 0.

A Banach space said satisfy Opial's condition [12] if $a_n \rightharpoonup a$ and $a \neq b$implying that:

$$\limsup_{n\to\infty}\|a_n - a\| < \limsup_{n\to\infty}\|a_n - b\|.$$

Let C be a nonempty closed convex subset of a real uniformly convex Banach space X; recall the mapping S_i (i=1,2,3,4) on C are said to satisfy condition (C) [15] if there exist a nondecreasing function f:[0,∞) → [0, ∞) with f (0) = 0 and f (r)> 0 for all r > 0 such that for all x ∈ C:

$max\{\|a - S_1a\|, \|a - S_2a\|, \|a - S_3a\|, \|a - S_4a\|\} \geq$ f $(D(a, \text{F}))$,

Where F = F(S_1) ∩ F(S_2) ∩ F(S_3) ∩ F(S_4), F(S_i)(i=1,2,3) are the sets of fixed points of S_i and$D(a$, F) = inf {‖a-q‖: q∈ F}.

Let C be a subset of a metric space (X, D). A mapping S: C → C is said to be semi-compact [14]if for a sequence $\{a_n\}$ in C with $\lim_{n\to\infty} D(a_n, Sa_n) = 0$, there exist a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $a_{n_j} \to p \in$ C.

Let C be nonempty subset of a normed space X and let G= (V(G), E(G)) be a directed graph such that V(G) = C. Then C is said to have property (G) [15] if for each sequence in C which is converging weakly to a∈ C and (a_n, a_{n+1})∈ E(G), there is a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that (a_{n_j}, x) ∈E(G) for j ∈ ℕ.

Remark 1: Since this paper we assume G is transitive then property(G) is equivalent to the property: if $\{a_n\}$ is a sequence in C with (a_n, a_{n+1})∈ E(G) such that for any subsequence $\{a_{n_j}\}$ of the sequence $\{a_n\}$ converging weakly to a in X, then $(a_n, a) \in$ E(G) for all n∈ ℕ.

In the consequence, the following results are needed to prove our main results.

Result 2.1 [15] Suppose that X is a Banach space having Opial's condition, C has property (G), and let S: C → C be a G-nonexpansive mapping. Then, 1-S is G-demiclosed at 0. i.e., if $a_n \rightharpoonup a$and a_n-S$a_n \to 0$, then a∈ F(S), where F(S) is the set fixed points of S.

Result 2.2 [17] Let $\{a_n\}$ and $\{b_n\}$ be two distinct sequences of nonnegative real numbers satisfying the inequality:

$a_{n+1} \leq a_n + b_n$ for all $n \geq 1$.

if$\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$exists.

Result 2. 3[13] Let X be a uniformly convex Banach space and $\{\alpha_n\}$ be a sequence in [λ, 1-λ] for someλ∈ (0,1). Assume that sequences $\{a_n\}$and $\{b_n\}$ in X are such that $\limsup_{n\to\infty} \| a_n \| \leq k$ and $\limsup_{n\to\infty} \| b_n \| \leq k$ and $\limsup_{n\to\infty} \| \alpha_n a_n + (1-\alpha_n)b_n \| = k$ for some $k \geq 0$. Then, $\lim_{n\to\infty} \| a_n - b_n \| = 0$.

Result 2.4[16] Let X be a Banach space which satisfies Opial's conditions and let $\{a_n\}$ be a sequence in X. Let x, y∈ X be such that $\lim_{n\to\infty} \| a_n - x \|$ and $\lim_{n\to\infty} \| a_n - y \|$ exist. If $\{a_{n_j}\}$ and $\{a_{n_k}\}$ are sub sequences of $\{a_n\}$ such that $a_{n_j} \rightharpoonup x$ and $a_{n_k} \rightharpoonup y$, then x=y.

3. Main results

Throughout the section, we let C be a nonempty closed convex subset of a Banach space X endowed with a directed graph G such that V(G) = C and E(G) is convex. We also suppose that G is transitive. The mapping S_i (i=1,2,3,4) are G-nonexpansive from C to C withF = $F(S_1) \cap F(S_2) \cap F(S_3) \cap F(S_4)$ nonempty. Then the sequence $\{a_n\}$ defined by (1.5)for an arbitrary $a_0 \in$ C is converges to a common fixed point of S_1, S_2, S_3 and S_4.

To prove our main results first we will show that:

Lemma 3.1 Let r_0∈ F be such that (a_0, r_0), (r_0, a_0) are in E(G). Then, (a_n, r_0), (b_n, r_0), (c_n, r_0), (d_n, r_0),(r_0, a_n), (r_0, b_n), (r_0, c_n), (r_0, d_n), (a_n, b_n), (a_n, c_n), (a_n, d_n) and (a_n, a_{n+1}) are in E(G).

Proof: We start with induction. Since $S_4$is edge preserving and (a_0, r_0) ∈ E(G) we have (S_4a_0, r_0) ∈ E(G) and so by convexity of E(G) we have (d_0, r_0), Again by convexity of E(G) withedge preserving of $S_3$and (d_0, r_0) ∈ E(G), we have (S_3d_0, r_0)∈E(G) we get(c_0, r_0)∈ E(G). Again, by convexity of E(G) and edge preserving of $S_2$and (c_0, r_0) ∈ E(G), we have (S_2c_0, r_0)∈ E(G) we get (b_0, r_0) ∈ E(G). Then by edge preserving of S_1 we have (S_1b_0, r_0)∈ E(G), and by convexity of E(G) we get (a_1, r_0) ∈ E(G). Thus, since S_4 is edge preservingand (a_1,r_0)∈ E(G), we have (S_4a_1, r_0) ∈ E(G) again by convexity of E(G) and (a_1, r_0),(S_4a_1),∈ E(G) we get (d_1, r_0) ∈E(G). Again since $S_3$is edge preserving and(d_1, r_0)∈ E(G), we have (S_3d_1, r_0) ∈ E(G), with the convexity of E(G) and $(d_1, r_0)$$(S_3d_1, r_0)$ ∈ E(G) we get (c_1, r_0) ∈ E(G). Then since S_2 is edge preserving and (c_1, r_0)∈ E(G), we have (S_2c_1, r_0) ∈ E(G), with the convexity of E(G) and $(c_1, r_0)$$(S_2c_1, r_0)$ ∈ E(G) we get(b_1, r_0) ∈ E(G). And hence (S_1b_1, r_0) ∈ E(G).

Next, we assume that (a_k, r_0) ∈ E(G). Since S_4 is edge preserving, we have (S_4a_k, r_0) and so by the convexity of E(G) and (a_k, r_0), (S_3a_k, r_0) ∈E(G) we get (d_k, r_0) ∈E(G). Again, since S_3 isedge preserving, (d_k, r_0)∈ E(G),we have(S_3d_k,r_0)∈ E(G), and by convexity of E(G)we get (c_k, r_0) ∈ E(G). Again, since S_2 is edge preserving, (c_k, r_0)∈ E(G), we have(S_2c_k, r_0)∈ E(G), and by convexity of E(G)we get (b_k, r_0) ∈ E(G). Then, since S_1 is edge preserving, (b_k, r_0)∈ E(G), we have (S_1b_k, r_0)∈E(G), and by convexity of E(G)we get(a_{k+1}, r_0) ∈ E(G).Thus, since S_4 is edge preserving and (a_{k+1}, r_0)∈ E(G), we have(S_4a_{k+1}, r_0) ∈E(G) again by convexity of E(G) and (a_{k+1}, r_0), (S_4a_{k+1},r_0) ∈ E(G) we get (d_{k+1}, r_0) ∈E(G). Again, since$S_3$is edge preserving and (d_{k+1}, r_0)∈ E(G), we have (S_3d_{k+1},r_0)∈ E(G). With the convexity of E(G) and (S_3d_{k+1}, r_0), (d_{k+1}, r_0)∈ E(G) we obtain (c_{k+1},r_0) ∈ E(G). Again, since S_2 is edge preserving and (c_{k+1}, r_0)∈ E(G), we have(S_2c_{k+1}, r_0)∈ E(G), With the convexity of E(G) and (S_2c_{k+1}, r_0), (c_{k+1}, r_0)∈ E(G) we obtain (b_{k+1},r_0) ∈ E(G). Therefore (a_n, r_0), (b_n, r_0), (c_n, r_0), (d_n, r_0)∈ E(G) for all $r \geq 1$. Since S_4 is edge preserving and (r_0, a_0) ∈E(G),we have (r_0, S_4a_0) ∈E(G) and so by convexity of E(G) (r_0, d_0) ∈ E(G). Again, sinceS_3 is edge preserving and (r_0, d_0)∈E(G), we have (r_0, S_3d_0) ∈E(G) by the convexity of E(G) we obtain (r_0, c_0) ∈E(G). Again, since S_2 is edge preserving and (r_0, c_0)∈E(G), we have (r_0, S_2c_0) ∈E(G) by the convexity of E(G) we obtain (r_0, b_0) ∈E(G). Using a similar process, we can show that(r_0, a_n), (r_0, b_n), (r_0, c_n), (r_0, d_n)∈E(G). By transitivity in E(G), we get (a_n, b_n), (a_n, c_n), (a_n, d_n) and (a_n, a_{n+1})∈ E(G). Thiscompletes the proof.

Lemma 3.2 let X be a uniformly convex Banach space. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$,$\{\gamma_n\}$ and $\{\delta_n\}$are real sequences in [λ,1-λ] for some δ ∈ (0, 1) and (a_0, r_0), (r_0, a_0) ∈ E(G) for arbitrary $a_0 \in$ C and r_0∈ E(G). Then,

(a) $\lim_{n\to\infty}\|a_n - r_0\|$ exists.
(b) $\lim_{n\to\infty}\|a_n - S_1a_n\| = \lim_{n\to\infty}\|a_n - S_2a_n\| = \lim_{n\to\infty}\|a_n - S_3a_n\| = \lim_{n\to\infty}\|a_n - S_4a_n\|$=0.

Proof: Let r_0∈ F by lemma 3.1,we have (a_n, r_0), (b_n, r_0), (c_n, r_0), (d_n, r_0), (a_n, b_n), (a_n, c_n), (a_n, d_n)∈ E(G), then by G-nonexpansiveness ofS_i (i =1, 2, 3,4) and using (2.4), we have:

$$\|d_n - r_0\| = \|(1-\delta_n)a_n + \delta_n S_4 a_n - r_0\|$$

$= \|(1-\delta_n)(a_n - r_0) + \delta_n(S_4 a_n - r_0)\|$
$\leq (1-\delta_n)\|a_n - r_0\| + \delta_n\|S_4 a_n - r_0\|$
$\leq (1-\delta_n)\|a_n - r_0\| + \delta_n\|a_n - r_0\|$
$\leq \|a_n - r_0\|,$ (3.1)

$\|c_n - r_0\| = \|(1-\gamma_n)d_n + \gamma_n S_3 d_n - r_0\|$

$= \|(1-\gamma_n)(d_n - r_0) + \gamma_n(S_3 d_n - r_0)\|$
$\leq (1-\gamma_n)\|d_n - r_0\| + \gamma_n\|S_3 d_n - r_0\|$
$\leq (1-\gamma_n)\|d_n - r_0\| + \gamma_n\|d_n - r_0\|$
$\leq \|d_n - r_0\|,$ (3.2)

$\|b_n - r_0\| = \|(1-\beta_n)c_n + \beta_n S_2 c_n - r_0\|$
$= \|(1-\beta_n)(c_n - r_0) + \beta_n(S_2 c_n - r_0)\|$
$\leq (1-\beta_n)\|c_n - r_0\| + \beta_n\|(S_2 c_n - r_0)\|$
$\leq (1-\beta_n)\|c_n - r_0\| + \beta_n\|(c_n - r_0)\|$
$\leq \|c_n - r_0\|,$ (3.3)

And also,

$\|a_{n+1} - r_0\| = \|(1-\alpha_n)b_n + \alpha_n S_1 b_n - r_0\|$
$= \|(1-\alpha_n)(b_n - r_0) + \alpha_n(S_1 b_n - r_0)\|$
$\leq (1-\alpha_n)\|b_n - r_0\| + \alpha_n\|(S_1 b_n - r_0)\|$
$\leq (1-\alpha_n)\|b_n - r_0\| + \alpha_n\|(b_n - r_0)\|$
$\leq \|b_n - r_0\| \leq \|a_n - r_0\|.$ (3.4)

It follows from result (2) that $\lim_{n\to\infty}\|a_n - r_0\|$ exists.In particular, the sequence $\{a_n\}$ is bounded.

(iii) Suppose that $\lim_{n\to\infty}\|a_n - r_0\| = k$. If k=0, then by G-nonexpansiveness of S_i (i =1, 2, 3,4), we get:

$\|a_n - S_i a_n\| \leq \|a_n - r_0\| + \|r_0 - S_i a_n\|$
$\leq \|a_n - r_0\| + \|r_0 - a_n\|.$

Therefore, the result follows. Assume that k> 0. Using (1.6) andtaking lim sup on both sides of inequality (1.8),we obtain:
$\lim_{n\to\infty}\sup\|b_n - r_0\| \leq \lim_{n\to\infty}\sup\|a_n - r_0\| = k.$(3.5)
By the G-nonexpansiveness of S_1, we have
$\|S_1 b_n - r_0\| \leq \|b_n - r_0\|.$
Taking lim sup on above inequality and using (3.5), we get
$\lim_{n\to\infty}\sup\|S_1 b_n - r_0\| \leq k.$(3.6)
Using (2.4) and taking lim sup on both sides of inequality (3.2), we have
$\lim_{n\to\infty}\sup\|c_n - r_0\| \leq \lim_{n\to\infty}\sup\|a_n - r_0\| = k.$ (3.7)
By using (3.7) and G-nonexpansiveness of S_2, we get
$\lim_{n\to\infty}\sup\|S_2 c_n - r_0\| \leq k.$(3.8)
Taking lim sup on both sides of inequality (3.1), we have
$\lim_{n\to\infty}\sup\|d_n - r_0\| \leq \lim_{n\to\infty}\sup\|a_n - r_0\| = k.$(3.9)
By using (3.9) and G-nonexpansiveness of S_3, we get

$$\lim_{n\to\infty}\sup\|S_3 d_n - r_0\| \leq k.$$

Since $\lim_{n\to\infty}\|a_n - r_0\|$ =k, taking limit sup both sides of inequality (3.4), we get
$\lim\sup_{n\to\infty}\|a_{n+1} - r_0\| = \lim\sup_{n\to\infty}\|(1-\alpha_n)(b_n - r_0) + \alpha_n(S_1 b_n - r_0)\|$=k. (3.10)
By using (3.5), (3.6),(3.10) and result 2.3, we get:
$\lim_{n\to\infty}\|S_1 b_n - b_n\| = 0.$ (3.11)
And also notice that, $\|a_{n+1} - r_0\| \leq \|b_n - r_0\|$.
Gives that:
$k \leq \lim_{n\to\infty}\inf\|b_n - r_0\|$.(3.12)

From (3.5) and (3.12), we have:
$\lim_{n\to\infty}\|b_n - r_0\| = k.$ (3.13)

By using (2.4) and (3.13), we have:
$\lim_{n\to\infty}\|(1-\beta_n)(c_n - r_0) + \beta_n(S_2c_n - r_0)\| = \text{k}.$ (3.14)
By (3.7), (3.8), (3.14) and result 2.3, we get:

$\lim_{n\to\infty}\|S_2c_n - c_n\| = 0.$ (3.15)
From (3.3) and (3.4), we have:

$$\|a_{n+1} - r_0\| \le \|c_n - r_0\|.$$

This implies that:
$\text{k} \le \lim_{n\to\infty}\inf\|c_n - r_0\|.$ (3.16)

From (3.7) and (3.16), we get:

$$\text{k} = \lim_{n\to\infty}\|c_n - r_0\|.$$

From the definition of c_n, we have:
$\lim_{n\to\infty}\|(1-\gamma_n)(d_n - r_0) + \gamma_n(S_3d_n - r_0)\|.$ (3.17)
By (3.9), (3.10), (3.17) and result 2.3, we get:
$\lim_{n\to\infty}\|S_3d_n - d_n\| = 0.$ (3.18)
From (3.2) and (3.3), we have:
$\|a_{n+1} - r_0\| \le \|d_n - r_0\|.$
This implies that:
$\text{k} \le \lim_{n\to\infty}\inf\|d_n - r_0\|.$ (3.19)
From (3.9) and (3.19) we have:

$$\text{k} = \lim_{n\to\infty}\|d_n - r_0\|.$$

This implies that:

$\text{k} = \lim_{n\to\infty}\|(1-\delta_n)(a_n - r_0) + \delta_n(S_4a_n - r_0)\|.$ (3.20)
Also, $\lim_{n\to\infty}\sup\|S_4a_n - r_0\| \le \lim_{n\to\infty}\sup\|a_n - r_0\| = \text{k}$, using (3.20) and result 2.3, we get:
$\lim_{n\to\infty}\|S_4a_n - a_n\| = 0.$ (3.21)
In addition,we see that:

$$\begin{aligned}\|d_n - a_n\| &= \|(1-\delta_n)a_n + \delta_n S_4a_n - a_n\| \\ &= \|(1-\delta_n)(a_n - a_n) + \delta_n(S_4a_n - a_n)\| \\ &= \delta_n\|(S_4a_n - a_n)\|.\end{aligned}$$

By (3.21), we have:
$\lim_{n\to\infty}\|d_n - a_n\| = 0.$ (3.22)
And by using (3.18), we have:

$$\begin{aligned}\|c_n - d_n\| &= \|(1-\gamma_n)d_n + \gamma_n S_3d_n - d_n)\| \\ &= \|(1-\gamma_n)(d_n - d_n) + \gamma_n(S_3d_n - d_n)\| \\ &= \gamma_n\|(S_3d_n - d_n)\| \\ &\to 0 \text{ as } n\to\infty. \qquad (3.23)\end{aligned}$$

Also by using (3.15), we have:

$$\begin{aligned}\|b_n - c_n\| &= \|(1-\beta_n)c_n + \beta_n S_2c_n - c_n)\| \\ &= \|(1-\beta_n)(c_n - c_n) + \beta_n(S_2c_n - c_n)\| \\ &= \beta_n\|(S_2c_n - c_n)\| \\ &\to 0 \text{ as } n\to\infty. \qquad (3.24)\end{aligned}$$

From (3.22) and (3.23), we have:

$$\begin{aligned}\|a_n - c_n\| &= \|a_n - d_n + d_n - c_n\| \\ &= \|a_n - d_n\| + \|d_n - c_n\| \\ &\to 0 \text{ as } n\to\infty. \qquad (3.25)\end{aligned}$$

And by (3.24) and (3.25), we have:
$\|a_n - b_n\| = \|a_n - c_n + c_n - b_n\|$

$$=\|a_n - c_n\| + \|c_n - b_n\|$$
$\to 0$ as $n \to \infty$. (3.26)

By using (3.18) and (3.22) together with G-nonexpansiveness of S_3,
we get:

$$\begin{aligned}\|S_3a_n - a_n\| &= \|S_3a_n - S_3d_n + S_3d_n - a_n\| \\ &\le \|S_3a_n - S_3d_n\| + \|S_3d_n - a_n\| \\ &\le \|a_n - d_n\| + \|S_3d_n - d_n\| + \|d_n - a_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \quad (3.27)$$

And by using (3.22), (3.23) and (3.25) together with G-nonexpansiveness of S_2, we have:

$$\begin{aligned}\|S_2a_n - a_n\| &= \|S_2a_n - d_n + d_n - a_n\| \\ &\le \|S_2a_n - d_n\| + \|d_n - a_n\| \\ &\le \|S_2a_n - S_2c_n + S_2c_n - d_n\| + \|S_3d_n - a_n\| \\ &\le \|S_2a_n - S_2c_n\| + \|S_2c_n - d_n\| + \|S_3d_n - a_n\| \\ &\le \|a_n - c_n\| + \|c_n - d_n\| + \|d_n - a_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \quad (3.28)$$

Also, by using (3.26)together with G-nonexpansiveness of S_1, we have:

$$\begin{aligned}\|S_1a_n - a_n\| &= \|S_1a_n - S_1b_n + S_1b_n - a_n\| \\ &\le \|S_1a_n - S_1b_n\| + \|S_1b_n - a_n\| \\ &\le \|S_1a_n - S_1b_n\| + \|S_1b_n - b_n + b_n - a_n\| \\ &\le \|a_n - b_n\| + \|b_n - b_n\| + \|b_n - a_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \quad (3.29)$$

Therefore, we obtain that $\lim_{n\to\infty}\|a_n - S_1a_n\| = \lim_{n\to\infty}\|a_n - S_2a_n\| = \lim_{n\to\infty}\|a_n - S_3a_n\| = \lim_{n\to\infty}\|a_n - S_4a_n\| = 0$. This completes the proof.

Lemma 3.3 Let X be a Banach space satisfying the Opial's condition and let C be a nonempty closed convex subset of X having property (G). Let G= (V(G), E(G)) be a directed graph with V (G)= C. Let S: C → C be G-nonexpansive mapping. Then I – S is demiclosed at 0.

Proof Assume that $\{a_n\}$ is a sequence in C which converges weakly to r, $(a_n, a_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $(I - S)a_n \to 0$. Since C has property (G), there is a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that$\{a_{n_j}, r\} \in E(G)$ for all $j \in \mathbb{N}$. Since $(I - S)a_n \to 0$, we have

$$\lim_{n\to\infty} \left\|a_{n_j} - Sa_{n_j}\right\| = 0$$

Suppose that $r \neq Sr$. Then by the Opial's condition, we get

$$\begin{aligned}\limsup_{n\to\infty} \left\|a_{n_j} - r\right\| &< \limsup_{n\to\infty} \left\|a_{n_j} - Sr\right\| \\ &\le \limsup_{n\to\infty} \left\|a_{n_j} - Sa_{n_j}\right\| + \left\|Sa_{n_j} - Sr\right\| \\ &\le \limsup_{n\to\infty} \left\|a_{n_j} - r\right\|,\end{aligned}$$

Which is a contradiction. This implies that $(I - S)a = 0$.

S_1, S_2, S_3 and S_4 are selfmappings in C with $F = F(S_1) \cap F(S_2) \cap F(S_3) \cap F(S_4) \neq \emptyset$ are said to satisfy the condition (C) if there is a nondecreasing function f: $[0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$max\{\|a - S_1a\|, \|a - S_2a\|, \|a - S_3a\|, \|a - S_4a\|\} \ge f(D(x, F)),$$

for all $x \in C$, where $D(a, F) = \inf_{b\in F} D(a, b)$. A mapping S: C→C is called

1. demicompact if any bounded sequence $\{a_n\}$ in C such that $\{a_n - Sa_n\}$ converges has a convergent subsequence;

(2) semicompact if any bounded sequence $\{a_n\}$ in C such that $\|a_n - Sa_n\| \to 0$ as $n \to \infty$has a convergent subsequence.

We see that every demicompact mapping is semicompact, but converse is not true in general.

Theorem 3.1 Let X be a real uniformly convex Banach space and C be a nonempty closed convex subset of X, S_i(i= 1,2,3,4) satisfying condition (C). Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are real sequences in [λ,1– λ]for some λ∈(0, 1),F is dominated by a_0 and F dominates a_0. Then, $\{a_n\}$ converges strongly to common fixed point of S_1, S_2, $S_3$and S_4.

ProofBy Lemma 3.1, we have $\lim_{n\to\infty}\|x_n - r\|$ exists and so $\lim D_{n\to\infty}(a_n, F)$ exists for any $r\in$ F. Also by Lemma 3.2 we have $\lim_{n\to\infty}\|a_n - S_1a_n\| = \lim_{n\to\infty}\|a_n - S_2a_n\| = \lim_{n\to\infty}\|a_n - S_3a_n\| = \lim_{n\to\infty}\|a_n - S_4a_n\| = 0$. It follows from the condition (C) that $\lim (D_{n\to\infty}(a_n, F))=0$. Since f: [0, ∞) → [0, ∞) is a nondecreasing function with f(0) = 0 and f(r) >0 for all$r\in(0, \infty)$. We obtain $\lim D_{n\to\infty}(a_n, F) = 0$.Hence we can find a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ and a subsequence $\{u_j\}$ contained in F such that $\left\|a_{n_j} - u_j\right\| \leq \frac{1}{2_j}$. Put $a_{j+1} = a_{j+k}$ for some k ≥1 we get,

$$\left\|a_{n_j} - u_j\right\| \leq \left\|a_{n_{j+k-1}} - u_j\right\| \leq \left\|a_{n_j} - u_j\right\| \leq \frac{1}{2_j}.$$

We obtain that $\|u_{j+1} - u_j\| \leq \frac{3}{2^{j+1}}$, so $\{u_j\}$ is a Cauchy sequence. Suppose that $u_j \to r_0 \in C$ as$j\to\infty$. Since F is closed, we get $r_0 \in$ F so we have $a_{n_j} \to r_0$ as $j \to \infty$. Since $\lim_{n\to\infty}\|a_n - r_0\|$ exists, we obtain $a_n \to r_0$. This completes the proof.

Theorem 3.2 Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X having the property (G), $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$are real sequences in [λ,1– λ] for some λ ∈ (0, 1), F is dominated by $a_0$and F dominates a_0. If one of S_i(i=1,2,3,4) is semi-compact, then$\{a_n\}$ converges strongly to common fixed point of S_1, S_2, S_3, and S_4.

Proof It follows from Lemma 3.2 and Lemma 3.3 that$\{a_n\}$ is bounded and $\lim_{n\to\infty}\|a_n - S_1a_n\| = \lim_{n\to\infty}\|a_n - S_2a_n\| = \lim_{n\to\infty}\|a_n - S_3a_n\| = \lim_{n\to\infty}\|a_n - S_4a_n\| = 0$.Since one of S_i(i= 1,2,3,4) is semi-compact, then there exist a sub sequence $\{a_{n_j}\}$ of $\{a_n\}$ such that$a_{n_j}\to r\in$ C as $j\to\infty$. Since C has property (G) and graph Gis transitive, we have (a_{n_j},r)∈ E(G). Also notice that, for each i∈ (1,2,3,4) $\lim_{n\to\infty}\left\|a_n - S_ia_{n_j}\right\| = 0$. Then:

$$\|r - S_ir\| \leq \left\|r - a_{n_j}\right\| + \left\|a_{n_j} - S_ia_{n_j}\right\| + \left\|S_ia_{n_j} - S_ir\right\|$$
$$\leq \left\|r - a_{n_j}\right\| + \left\|a_{n_j} - S_ia_{n_j}\right\| + \left\|a_{n_j} - r\right\|$$
$$\to 0 \text{ (as} j\to\infty).$$

Hence, $r\in$ F. Thus, $\lim_{n\to\infty}(a_n, F)$ exists by theorem 3.1. We note that$\lim D_{n\to\infty}(a_{n_j}, F) \leq \lim D_{n\to\infty}(a_{n_j}, r)\to 0$ as $j\to\infty$. Hence $\lim_{n\to\infty}(a_n, F) = 0$. It follows, as in the proof of Theorem 3.1, that $\{a_n\}$ converges strongly to a common fixed of S_1, S_2, S_3 and S_4. This completes the proof.

Theorem 3.3 Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X having the property (G), Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are real sequences in [λ,1– λ] for some λ∈ (0, 1),If (a_0, r_0), (r_0a_0) ∈ E(G) for arbitrary $a_0\in$ Cand $r_0\in$F, then $\{a_n\}$ converges weakly to a common fixed of S_1, S_2, S3 and S_4.

Proof Let $r_0 \in$ F be such that (a_0, r_0), (r_0a_0) ∈ E(G). From Lemma 3.1, we have $\lim_{n\to\infty}\|a_n - r_0\|$ exists, so $\{a_n\}$ is bounded. It follows from Lemma 3.2 that$\lim_{n\to\infty}\|a_n - S_1a_n\| = \lim_{n\to\infty}\|a_n - S_2a_n\| = \lim_{n\to\infty}\|a_n - S_3a_n\| = \lim_{n\to\infty}\|a_n - S_4a_n\| = 0$.Since$\{a_n\}$ is bounded uniformly convex, we may assume that $a_n\to x$as $n \to \infty$, without loss of generality. By result 2.1, we have $x\in$ F. Suppose that subsequences $\{a_{n_j}\}$ and $\{a_{n_k}\}$ of $\{a_n\}$ converge weakly to $x$and y, respectively, By Lemma 3.2, we obtain that $\left\|a_{n_j} - S_ia_{n_j}\right\|\to 0$, $\left\|a_{n_k} - S_ia_{n_k}\right\| \to 0$

as $j, k \to \infty$. Using Result 3.1 we have $x, x \in$ F. By Lemma 3.1 $\lim_{n\to\infty}\|a_n - x\|$and $\lim_{n\to\infty}\|a_n - y\|$ exist. It follows from Result 5 that $x = y$. Therefore, $\{a_n\}$ converges weakly to a common fixed of S_1, S_2, S_3 and S_4 This completes the proof.

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