QUASI IDEALS & BI-IDEALS IN TERNARY Γ-SO-SEMIRINGS

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Abstract: "The set of all partial functions over a set under a natural addition, functional composition and functional relation on the, forms a Γ-SO-ring. The concepts of quasi ideals, minimal quasi ideals, prime bi-ideal, semi prime bi-ideals in ternary Γ-SO-ring are introduced."

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1. Introduction:

IBI.Irreduciblesemiprime bi-ideal is denoted by ISPBI".

2. Prerequisites:

The following are the prerequisites for this paper."The notion of bi-ideal is introduced by Good and Hughes. Bi-ideals in Ternary semigroups are studied by Dixit and Dewan. In 1981 the concept of Γ - semigroup as generalization of semi group introduced by Sen. H.S Vandiver develops the theory of semi ring in 1934. The notion of Γ - semirings was introduced by M.MuralikrishnaRao in 1995.

Some classical notion of ternary Γ - SO semirings are introduced in this paper.In 2019 K.Bhagyalakshmi and Dr.V.AmarendraBabu developed ideal theory in Ternary Γ-SO semirings.In this paper we introduce the notions of irreducible, strongly irreducible bi-ideals of Ternary Γ-SO semirings and obtain characterizations of prime, semiprime, irreducible and strongly irreducible bi-ideals in regular Ternary Γ-SO semiring. Throughout this paper Ternary Γ-SO semiring is denoted by TΓSS, Complete Ternary Γ-SO semiring is denoted by CTΓSS, bi-ideal is denoted by BI, prime ideal by PI, prime bi-ideal is denoted by PBI, semiprime bi-ideal is denoted by SPBI, Irreducible bi-ideal is denoted by IBI.

Definition2.1: "A be ***partial Γ-monoid*** is a triple (R, Γ, Σ) where R, Γ are non-empty sets and Σ is a partial addition defined on some but not necessarily all families $(a_i : i \in I)$ in R with the following laws:

1) **Unary sum axiom:** If $(a_i : i \in I)$ is a one element family in R and I = {*j*} then $\sum(a_i : i \in I)$ is defined and equal to a_j.
2) **Partition Associative axiom:** If $(a_i : i \in I)$ is a family in R and $(a_j : j \in I)$ is a partition of I, then $(a_i : i \in I)$ is sum-able if and only if $(a_i : i \in I_j)$ is sum-able for every j in J, $(\sum(a_i : i \in I_j) : j \in J)$ is sum-able and $\sum(a_i : i \in I_j) = \sum(\sum(a_i : i \in I_j) : j \in J)$."

Definition 2.2: "Let M, Γ be partial Γ-monoids then M is said to be ***partial ternary gamma semiring*** provided ∃ a mapping $M \times \Gamma \times M \times \Gamma \times M \to M$ satisfying the following conditions:

1) $x\alpha y\beta(z\delta p\gamma q) = x\alpha(y\beta z\delta p)\gamma q = (x\alpha y\beta z)\delta p\gamma q$
2) a family $(a_i : i \in I)$ is sum-able in M implies that $(x\alpha y\beta a_i :$ for odd $i \in I)$ is sum-able in M and $x\alpha y\beta\left\lceil\sum(a_i : i \in I)\right\rceil = \sum(x\alpha y\beta a_i :$ for odd $i \in I)$
3) family $(a_i : i \in I)$ is sum able in M implies that $(x\alpha a_i\beta y :$ for odd $i \in I)$ is sum able in M and $x\alpha\left\lceil\sum(a_i : i \in I)\right\rceil\beta y = \sum(x\alpha a_i\beta$ y$:$ for odd $i \in I)$
4) family $(a_i : i \in I)$ is sum able in M implies that $(a_i\alpha x\beta y :$ for odd $i \in I)$ is sum able in M and $\left\lceil\sum(a_i : i \in I)\right\rceil\alpha x\beta y = \sum(a_i\alpha x\beta y :$ for odd $i \in I)$"

Definition 2.3: "A partial ternary Γ-semiring said to have a left (lateral, right) unity element provided there exist a family $(e_i : i \in I)$ of M and $(\alpha_i, \beta_i : i \in I)$ of Γ ∋
$\sum e_i\alpha_i e_i\beta_i a = a(\sum e_i\alpha_i a\beta_i e_i = a, \sum a\alpha_i e_i\beta_i e_i = a)$ for any $a \in$ M."
Definition 2.4: "The sum ordering relation ≤ in partially ternary Γ-monoid M is the binary relation such that $a \le b$ iff there exist an element c in M such that $b = a + c\ \forall\ a, b \in$ M."
Definition 2.5: "A ***sum ordered partially ternary* Γ *-monoid (ternary* Γ *-so-monoid)*** in which partial sum ordering is a partial ordering."
Definition 2.6: "A partial ternary Γ-semiring M is said to be ***sum ordered partial ternary* Γ *-semiring (Ternary* Γ *-SO-semiring)*** if the partial Γ-monoid is SO-Γ-monoid."
Definition 2.7: "Let M be a partial ternary Γ-semiring. A non-empty subset of M is said to be ***left (Lateral, right) partial ternary Γ-ideal*** of M provided
(i) $(a_i : i \in I)$ is a sum able family of M and $x_i \in$ A for all $i \in$ I implies $\sum_i x_i \in A$
(ii) for all $x, y \in$ M, $z \in$ A implies that $z\alpha x\beta y \in$ A ($x\alpha z\beta y \in$ A, $x\alpha y\beta z \in$ A)

If A is left, lateral and right partial ternary Γ-ideal of M, then A is called partial ternary Γ-ideal of M."
Definition 2.8: "Let M be a ternary Γ-SO-semiring. A non-empty subset A of M is said to be a ***left (lateral, right) ternary Γ-ideal*** of M, if it satisfies the following:
(i) A is a left (lateral, right) partial ternary Γ-ideal of M.
(ii) $x \in$ M and $y \in$ A such that $x \le y$ then $x \in$ A.
If A is left, lateral as well as right ternary Γ-ideal of M, then A is known as ternary Γ-ideal of M."
Definition 2.9: "Let M be a ternary Γ-SO-semiring and A be a subset of M, then the intersection of all ternary Γ-ideals containing the set A is called ***ternary Γ-ideal generated by A*** and it is denoted by (A)."

Definition 2.10: **"**A Ternary Γ-SO-semiring M is said to be ***complete ternary Γ-so-semiring*** if every family of elements in M is sum able."
Definition 2.11: **"**A proper ideal P of a ternary Γ-SO-semiring M is known as ***prime*** if and only if for any ideals R, S, T of M, RΓSΓT ⊆ P ⟹ R ⊆ P or S ⊆ P or T ⊆ P."
Definition2.12: **"**A proper ideal P of a ternary Γ-SO-semiring M is known as ***semiprime*** if and only if for any ideals R of M, RΓRΓR ⊆ P ⟹ R ⊆ P".
Definition2.13: **"**A non-empty subset A of a Γ-SO-ring R is said to be Γ-sub SO-ring if

(i) A is a sub-SO monoid of R
(ii) AΓAΓA⊆ A"

Definition2.14: A TΓSS N is said to be zero divisor free (ZDF) if for $l, m, n \in N, \alpha, \beta \in \Gamma$ $[l\alpha m\beta n] = 0 \Rightarrow$ l=0 or m=0 or n=0.
Definition2.15: A TΓSS N is said to be multiplicatively left Γ-cancellative (MLC) if $l\Gamma m\Gamma u = l\Gamma m\Gamma v$ implies that u=v for all $l, m, u, v \in N$
Definition2.16: A TΓSS N is said to be multiplicatively laterally Γ-cancellative (MLLC) if $l\Gamma u\Gamma m = l\Gamma v\Gamma m$ implies that u=v for all $l, m, u, v \in N$.
Definition2.17: A TΓSS N is said to be multiplicatively right Γ-cancellative (MRC) if $u\Gamma l\Gamma m = v\Gamma l\Gamma m$ implies that u=v for all $l, m, u, v \in N$
Definition2.18: A TΓSS N is called as multiplicatively Γ-cancellative (MC) if it is multiplicative left Γ-cancellative (MLC), multiplicative right Γ-cancellative (MRC)& multiplicative laterally Γ-cancellative (MLLC).
Theorem2.19: A multiplicative Γ-cancellative TΓSS N is zero divisorfree.

Quasi-Ideals in Ternary Γ-SO-semiring:

Definition3.01: An additive subsemigroup M of a TΓSS N is said to be a quasi ideal of N if $(M\Gamma N\Gamma N) \bigcap (N\Gamma M\Gamma N + N\Gamma N\Gamma M\Gamma N\Gamma N) \bigcap (N\Gamma N\Gamma M) \subseteq M$.

Note3.02: Every QI of a TΓSS N is a Ternary subsemiring of N.

Lemma3.03: "Every left, right & middle (or lateral) ideal of TΓSS N is a QI of N".

Remark3.04:"The converse of lemma 3.3 is not true; in general, that is, a quasi ternary Γ-ideal may not be a left, a right, or a lateral ternary Γ-ideal of N. This follows from the following example."

Example3.05:"Let $M_2(Z_0^-)$ be the TΓSS of the set of all 2X2 square matrices over Z_0^-, the set of all non positive integers and Γ be the set of all 2×2 square matrices over Z_0^-, the set of all negative integers. Let W={$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in Z_0^-$} then we can easily verify that W is a quasi ternary Γ-ideal of N, but W is not a right ternary Γ-ideal, a lateral Γ-ideal or left ternary Γ-ideal of N".

Theorem3.06: If M is a QI of TΓSS N & O is a ternary sub semiring of N then $N \bigcap O$ is a QI of O.

Lemma3.07: The arbitrary intersection of QI of a TΓSS N is a QI of N.

Theorem3.08: "An additive subsemigroup M of a TΓSS N is a QI of N if M is the intersection of right ideal, a lateral ideal & a left ideal of N".

Proof: Let I, J, K be a (right/lateral/left ideals), of N∋ $M = I \cap J \cap K$. By lemma 3.03 & lemma 3.07, we conclude that M is a QI of N.

The contrary of Th3.08 does not hold, in general. Then in particular the following result we have.

Theorem3.09:"An additive subsemigroup M of a TΓSS N is a minimal quasi ideal of N iff M is the intersection of a minimal right ideal, a minimal lateral ideal & a minimal left ideal of N".

Proof: Let I, J, K (minimal right/ minimal lateral/minimal left ideals) of N ∋ $M = I \cap J \cap K$.
Then by Th 3.8, it follows that M is a QI of N. Now it remains to P.T M is minimal. If possible let $M' \subseteq M$ be any other QI of N. Then $M'\Gamma N\Gamma N$ is a right ideal of N&
$M'\Gamma N\Gamma N \subseteq M\Gamma N\Gamma N \subseteq I\Gamma N\Gamma N \subseteq I$. ∵ I is a minimal right ideal of N, we have $M'\Gamma N\Gamma N = I$.
Similar we can Prove that $N\Gamma M'\Gamma N + N\Gamma N\Gamma M'\Gamma N\Gamma N = I$ & $N\Gamma N\Gamma M' = I$
$\therefore M = I \cap J \cap K \Rightarrow M'\Gamma N\Gamma N \cap (N\Gamma M'\Gamma N + N\Gamma N\Gamma M'\Gamma N\Gamma N) \cap N\Gamma N\Gamma M' \subseteq M'$.
Consequently, $M = M'$ & thus Q is a minimal QI of N.
Conversely, let M be a minimal QI of N then
$M\Gamma N\Gamma N \cap (N\Gamma M\Gamma N + N\Gamma N\Gamma M\Gamma N\Gamma N) \cap N\Gamma N\Gamma M \subseteq M$
Let $m \in M$ then $m\Gamma N\Gamma N$ is a right ideal, ($(N\Gamma m\Gamma N + N\Gamma N\Gamma m\Gamma N\Gamma N)$ is a middle ideal &
$N\Gamma N\Gamma m$ is a left ideal of N. ∴by theorem 7.8,
$m\Gamma N\Gamma N \cap (N\Gamma m\Gamma N + N\Gamma N\Gamma m\Gamma N\Gamma N) \cap N\Gamma N\Gamma m \subseteq M\Gamma N\Gamma N \cap (N\Gamma M\Gamma N + N\Gamma N\Gamma M\Gamma N\Gamma N) \cap N\Gamma N\Gamma M \subseteq M$
∵ M is a minimal quasi-ideal of N, we have
$m\Gamma N\Gamma N \cap (N\Gamma m\Gamma N + N\Gamma N\Gamma m\Gamma N\Gamma N) \cap N\Gamma N\Gamma m = M$
Now to show that $m\Gamma N\Gamma N$, $(N\Gamma m\Gamma N + N\Gamma N\Gamma m\Gamma N\Gamma N)$ & $N\Gamma N\Gamma m$ are respectively minimal right, a minimal lateral & a minimal left ideal of N. If possible, let I be any right ideal of N ∋
$I \subseteq m\Gamma N\Gamma N$ then $I\Gamma N\Gamma N \subseteq I \subseteq m\Gamma N\Gamma N$. Now
$I\Gamma N\Gamma N \cap (N\Gamma m\Gamma N + N\Gamma N\Gamma m\Gamma N\Gamma N) \cap N\Gamma N\Gamma m \subseteq m\Gamma N\Gamma N \cap (N\Gamma m\Gamma N + N\Gamma N\Gamma m\Gamma N\Gamma N) \cap N\Gamma N\Gamma m \subseteq M$
Thus by minimality of M, we find that $M = I\Gamma N\Gamma N \cap (N\Gamma m\Gamma N + N\Gamma N\Gamma m\Gamma N\Gamma N) \cap N\Gamma N\Gamma m$
$\Rightarrow M \subseteq I\Gamma N\Gamma N$, again $m\Gamma N\Gamma N \subseteq M\Gamma N\Gamma N \subseteq (I\Gamma N\Gamma N)\Gamma N\Gamma N \subseteq I\Gamma N\Gamma N$
Thus $m\Gamma N\Gamma N = I\Gamma N\Gamma N \subseteq I$ &thus $I = m\Gamma N\Gamma N$ .Consequently, $m\Gamma N\Gamma N$ is a minimal right ideal of N. Similarly we can prove that $(N\Gamma m\Gamma N + \Gamma N\Gamma N m\Gamma N\Gamma N)$ is a minimal lateral ideal and $N\Gamma N\Gamma m$ is a minimal left ideal of N.

Proposition3.10:"Any minimal lateral ideal of a TΓSS H is a minimal ideal of H".

Proof: Let Q be a minimal lateral ideal of H. We shall S.T Q is a minimal ideal of H. Let
Let $q \in Q$ then $(H\Gamma q\Gamma H + H\Gamma H\Gamma q\Gamma H\Gamma H)$ is a lateral ideal of H&
$(H\Gamma q\Gamma H + H\Gamma H\Gamma q\Gamma H\Gamma H) \subseteq (H\Gamma Q\Gamma H + H\Gamma H\Gamma Q\Gamma H\Gamma H) \subseteq Q$
∵ Q is minimal, we have $Q = (H\Gamma q\Gamma H + H\Gamma H\Gamma q\Gamma H\Gamma H)$ now
$Q\Gamma H\Gamma H = (H\Gamma q\Gamma H + H\Gamma H\Gamma q\Gamma H\Gamma H)\Gamma H\Gamma H = (H\Gamma q\Gamma H)\Gamma H\Gamma H + (H\Gamma H\Gamma q\Gamma H\Gamma H)\Gamma H\Gamma H$
$\subseteq H\Gamma q\Gamma H + H\Gamma H\Gamma q\Gamma H\Gamma H \subseteq Q$ & $H\Gamma H\Gamma Q = H\Gamma H\Gamma(H\Gamma q\Gamma H + H\Gamma H\Gamma q\Gamma H\Gamma H)$
$\subseteq H\Gamma q\Gamma H + H\Gamma H\Gamma q\Gamma H\Gamma H \subseteq Q$.

$\Rightarrow$ Q is a right ideal & also left ideal of N. Also Q is a lateral ideal of N. Thus Q is an ideal of N. To P.T Q is a minimal ideal of N. If possible, let Q' be an ideal of N∋ $Q' \subseteq Q$. $\because Q'$ is an ideal of N, it is a lateral ideal of N. By assumption, thus $Q' = Q$.

So Q is a minimal ideal of N.

Corollary3.11:"Any minimal quasi-ideal of a TΓSS N is contained in a minimal ideal of N.

Proof: Let M be a minimal quasi-ideal of N. Then by theorem 3.9 $M = I \bigcap J \bigcap K$, where I is a minimal right ideal, J is a minimal lateral ideal & L a minimal left ideal of N. Clearly $M \subseteq Q$ from Th.3.10, it follows that M is a minimal ideal of N".

Proposition3.12: Let g be an idempotent element of a TΓSS N. If I is a right ideal, J is a lateral ideal & K a left ideal of N then $I\Gamma g\Gamma g$, $g\Gamma g\Gamma J\Gamma g\Gamma g$ & $g\Gamma g\Gamma K$ are quasi ideals of N.

Proof: To show $I\Gamma g\Gamma g$, $g\Gamma g\Gamma J\Gamma g\Gamma g$ & $g\Gamma g\Gamma K$ are quasi-ideals of N, it is sufficient to show, $I\Gamma g\Gamma g = I \bigcap (N\Gamma g\Gamma N + N\Gamma N\Gamma g\Gamma N\Gamma N) \bigcap N\Gamma N\Gamma g$, $g\Gamma g\Gamma J\Gamma g\Gamma g = g\Gamma N\Gamma N \bigcap J \bigcap N\Gamma N\Gamma g$ & $g\Gamma g\Gamma K = g\Gamma N\Gamma N \bigcap (N\Gamma g\Gamma N + N\Gamma N\Gamma g\Gamma N\Gamma N) \bigcap K$ for the first case, it is clear that, $I\Gamma g\Gamma g \subseteq I \bigcap N\Gamma N\Gamma g$. Let $b \in I \bigcap N\Gamma N\Gamma g$ then $b \in I$ & $b \in N\Gamma N\Gamma g$. Now

$$b \in N\Gamma N\Gamma g \Rightarrow \sum_{i=1}^{n} s\alpha t\beta g \text{ for some } s,t \in N, \alpha,\beta,\gamma,\delta \in \Gamma$$

$$b\Gamma N\Gamma g = (\sum_{i=1}^{n} s\alpha t\beta g)\gamma g\delta g \text{ for some } s,t \in N, \alpha,\beta,\gamma,\delta \in \Gamma$$

$$= \sum_{i=1}^{n} s\alpha t\beta(g\gamma g\delta g) \text{ for some } s,t \in N, \alpha,\beta,\gamma,\delta \in \Gamma$$

$$= \sum_{i=1}^{n} s\alpha t\beta g = b$$ it follows that $b \in I\Gamma g\Gamma g$ & hence $I\Gamma g\Gamma g = I \bigcap N\Gamma N\Gamma g$. Again $b = b\Gamma g\Gamma g \in N\Gamma g\Gamma g$ & $O \in N\Gamma N\Gamma g\Gamma N\Gamma N$. So we find that $b \in N\Gamma g\Gamma g + N\Gamma N\Gamma g\Gamma N\Gamma N$.

Consequently, $I\Gamma g\Gamma g = I \bigcap (N\Gamma g\Gamma N + N\Gamma N\Gamma g\Gamma N\Gamma N) \bigcap N\Gamma N\Gamma g$ for the second case, we see that $g\Gamma g\Gamma J\Gamma g\Gamma g \subseteq g\Gamma N\Gamma N \bigcap J \bigcap N\Gamma N\Gamma g$. Let $b \in g\Gamma N\Gamma N \bigcap J \bigcap N\Gamma N\Gamma g$ then $b \in g\Gamma N\Gamma N, b \in J$ & $b \in N\Gamma N\Gamma g$ now $b \in g\Gamma N\Gamma N$ & $g \in N\Gamma N\Gamma g$

$$\Rightarrow b = \sum_{i=1}^{m} g\alpha_i s_i \beta_i t_i \text{ for some } s_i, t_i \in N, \alpha_i, \beta_i \in \Gamma$$

$$= \sum_{j=1}^{n} u_j \alpha_j v_j \beta_j g \text{ for some } u_j, v_j \in N, \alpha_j, \beta_j \in \Gamma$$

$$\therefore g\Gamma g\Gamma b\Gamma g\Gamma g = g\Gamma g\Gamma(\sum_{i=1}^{m} g\alpha_i s_i \beta_i t_i)\Gamma g\Gamma g$$

$$= (\sum_{i=1}^{m} (g\Gamma g\Gamma g)\alpha_i s_i \beta_i t_i)\Gamma g\Gamma g$$

$$= (\sum_{i=1}^{m} (g\alpha_i s_i \beta_i t_i)\Gamma g\Gamma g$$

$$= \sum_{j=1}^{n} (u_j \alpha_j v_j \beta_j g)\Gamma g\Gamma g$$

$= \sum_{j=1}^{n} u_j \alpha_j v_j \beta_j g$ =b

Consequently $b \in g\Gamma g\Gamma J\Gamma g\Gamma g$ & $g\Gamma g\Gamma J\Gamma g\Gamma g = g\Gamma N\Gamma N \bigcap J \bigcap N\Gamma N\Gamma g$

Theorem3.13:"If for every quasi ideal M of N, $M\Gamma M\Gamma M = M$ then N is a regular TΓSS".

Proof: If I, J, K is (minimal right / minimal / minimal left) ideals of N. Then by Th.3.9 it follows that $I \bigcap J \bigcap K$ is a quasi-ideal of N. Now by hypothesis,

$I \bigcap J \bigcap K = (I \bigcap J \bigcap K)\Gamma(I \bigcap J \bigcap K)\Gamma(I \bigcap J \bigcap K) \subseteq I\Gamma J\Gamma K$

Clearly, $I\Gamma J\Gamma K \subseteq I \bigcap J \bigcap K$. So $I\Gamma J\Gamma K = I \bigcap J \bigcap K$, by known theorem regular TΓSS

Lemma3.14: "Every quasi-ideal of a TΓSS Q is a bi-ideal of Q".

Proof: Let W be a Quasi-ideal of Q then we see that

$W\Gamma Q\Gamma Q\Gamma Q\Gamma W \subseteq W\Gamma(Q\Gamma Q\Gamma Q) \subseteq W\Gamma Q\Gamma Q$

$W\Gamma Q\Gamma W\Gamma Q\Gamma W \subseteq Q\Gamma(Q\Gamma Q\Gamma Q)\Gamma W \subseteq Q\Gamma Q\Gamma W$ &

$W\Gamma Q\Gamma W\Gamma Q\Gamma W \subseteq Q\Gamma Q\Gamma W\Gamma Q\Gamma Q$

Again $\{0\} \subseteq Q\Gamma Q\Gamma W \Rightarrow W\Gamma Q\Gamma W\Gamma Q\Gamma W \subseteq Q\Gamma Q\Gamma W + Q\Gamma Q\Gamma W\Gamma Q\Gamma Q$

Consequently, it follows that

$\Rightarrow W\Gamma Q\Gamma W\Gamma Q\Gamma W \subseteq W\Gamma Q\Gamma Q \bigcap (Q\Gamma W\Gamma Q + Q\Gamma Q\Gamma W\Gamma Q\Gamma Q) \bigcap Q\Gamma Q\Gamma W \subseteq W$

Thus W is a BI of Q.

Note3.15:"The converse of above lemma does not hold, in general that is a bi-ideal of a TΓSS N may not be a quasi-ideal of N."

Remark3.16: "Every left, right and lateral ideal of N is a quasi ideal of N, it follows that every left, right & lateral ideal of N is a bi-ideal of N but the converse is not true in general."

Proposition3.17: If D is a BI of a TΓSS N & U is a Ternary sub semiring of N, then $D \bigcap U$ is a BI of U.

Lemma3.18: If D is a BI of a TΓSS N & U_1, U_2 are two ternary subsemirings of N, then $D\Gamma U_1\Gamma U_2$ $U_1\Gamma D\Gamma U_2$ & $U_1\Gamma U_2\Gamma D$ are bi-ideals of N.

Corollary3.19: If D_1, D_2 & D_3 are three bi-ideals of a TΓSS N then $D_1\Gamma D_2\Gamma D_3$ is a bi-ideal of N.

Corollary3.20: If W_1, W_2 & W_3 are three quasi-ideals of a TΓSS N then $W_1\Gamma W_2\Gamma W_3$ is a BI of N.

In general, if D is a BI of a TΓSS N & E is a BI of D then E is not a BI of N. Particularly the result as follows.

Theorem3.21: **"**Let D be a bi-ideal of a TΓSS N & Gis a bi-ideal of D∋ $G\Gamma G\Gamma G = G$ then G is a bi-ideal of N".

Proof:∵ D is a BI of N, $D\Gamma N\Gamma D\Gamma N\Gamma D \subseteq D$ &∵ G is a bi-ideal of D, $G\Gamma D\Gamma G\Gamma D\Gamma G \subseteq G$

$\because G\Gamma N\Gamma G\Gamma N\Gamma G = (G\Gamma G\Gamma G)\Gamma N\Gamma G\Gamma N\Gamma(G\Gamma G\Gamma G)$

$= G\Gamma G\Gamma(G\Gamma N\Gamma G\Gamma N\Gamma G)\Gamma G\Gamma G \subseteq G\Gamma G\Gamma(D\Gamma N\Gamma D\Gamma N\Gamma D)\Gamma G\Gamma G \subseteq G\Gamma G\Gamma D\Gamma G\Gamma G$

$= G\Gamma G\Gamma D\Gamma G\Gamma(G\Gamma G\Gamma G) \subseteq G\Gamma(G\Gamma D\Gamma G\Gamma D\Gamma G)\Gamma G \subseteq G\Gamma G\Gamma G = G$

Thus G is a BI of N.

Theorem3.22: "A TΓSS N has no non-zero proper BI if N is a ternary division semiring".

Pf: Let D be a Ternary division semiring& B be a non-zero BI of D. Let

$b(\neq 0) \in B, \exists n(\neq 0) \in D \ni b\alpha n\beta y = n\alpha b\beta y = y\alpha b\beta n = y\alpha n\beta b = y \forall y \in D$

$\Rightarrow D = B\Gamma D\Gamma D = D\Gamma D\Gamma B$, now $\Rightarrow D = B\Gamma D\Gamma D = B\Gamma(D\Gamma D\Gamma B)\Gamma(D\Gamma D\Gamma B)$

$= B\Gamma(B\Gamma D\Gamma D)\Gamma(D\Gamma B\Gamma D)\Gamma(D\Gamma D\Gamma B)\Gamma B$

$\subseteq B\Gamma(B\Gamma D\Gamma B\Gamma D\Gamma B)\Gamma B \subseteq B\Gamma B\Gamma B \subseteq B$

So, B=D&therefore B has no non-zero proper bi-ideal.
The contraryof theabove Thneed not be true.
Theorem3.23:“A TΓSS N is a ternary division semiring if N is MC & has no non-zero proper bi-ideal”.
Pf: Let N be a MC TΓSS & has no non-zero proper bi-ideal. If $n(\neq 0)\in N$ then $n\Gamma N\Gamma y$ & $y\Gamma n\Gamma N$ are two bi-ideals of N for any non-zero $y\in N$. $\because$ N is MC, it is ZDF.
$\therefore$ $n\Gamma N\Gamma y\neq\{0\}$ & $y\Gamma n\Gamma N\neq\{0\}$. By hypothesis, we have $n\Gamma N\Gamma y=y\Gamma n\Gamma N=N$ & hence for $y(\neq 0)\in N,\exists b,c\in N\ni n\alpha b\beta y=y\alpha n\beta c=y$. Let c be any element of N, $\exists d,e\in N\ni$ $n\gamma d\delta y=y\gamma n\delta e=c$. Thus
$n\alpha b\beta c=n\alpha b\beta(y\gamma n\delta e)=(n\alpha b\beta y)\gamma n\delta e=y\gamma n\delta e=c\forall c\in N,\alpha,\beta,\gamma$ & $\delta\in\Gamma$
Similarly we can S.T $b\alpha n\beta=c\alpha b\beta n=c\forall c\in N$
$\therefore$ we find that $n\alpha b\beta c=c\alpha n\beta b=c\alpha b\beta n=c\forall c\in N$.
$\therefore$ N is a ternary division semiring.
Theorem3.24: “Let I, J & K be three ternary subsemirings of a TΓSS M& $D=I\Gamma J\Gamma K$ then D is a bi-ideal if at least one I, J, K is a right or a lateral or a left ideal of M”.
Proof: Let $D=I\Gamma J\Gamma K$. Suppose I is a right ideal of M.
Then we find that
$(I\Gamma J\Gamma K)\Gamma M\Gamma(I\Gamma J\Gamma K)\Gamma M\Gamma(I\Gamma J\Gamma K)=I\Gamma(M\Gamma M\Gamma M)\Gamma(M\Gamma M\Gamma M)\Gamma M\Gamma M\Gamma J\Gamma K$
$\subseteq I\Gamma(M\Gamma M\Gamma M)\Gamma M\Gamma J\Gamma K\subseteq(I\Gamma M\Gamma M)\Gamma J\Gamma K\subseteq I\Gamma J\Gamma K$
Thus $D=I\Gamma J\Gamma K$ is a bi-ideal of M.
Now presume that J is a right ideal of M. Then
$(I\Gamma J\Gamma K)\Gamma M\Gamma(I\Gamma J\Gamma K)\Gamma M\Gamma(I\Gamma J\Gamma K)\subseteq I\Gamma J\Gamma(M\Gamma M\Gamma M)\Gamma(M\Gamma M\Gamma M)\Gamma M\Gamma M\Gamma K$
$\subseteq I\Gamma J\Gamma(M\Gamma M\Gamma M)\Gamma M\Gamma K\subseteq I\Gamma(J\Gamma M\Gamma M)\Gamma K\subseteq I\Gamma J\Gamma K$
$\Rightarrow D=I\Gamma J\Gamma K$ is a bi-ideal of M.
Again, if K is a right ideal of M, then
$(I\Gamma J\Gamma K)\Gamma M\Gamma(I\Gamma J\Gamma K)\Gamma M\Gamma(I\Gamma J\Gamma K)\subseteq(I\Gamma J\Gamma K)\Gamma(M\Gamma M\Gamma M)\Gamma(M\Gamma M\Gamma M)\Gamma M\Gamma M$
$\subseteq(I\Gamma J\Gamma K)\Gamma(M\Gamma M\Gamma M)\Gamma M\subseteq I\Gamma J\Gamma(K\Gamma M\Gamma M)\subseteq I\Gamma J\Gamma K$
Consequently, $D=I\Gamma J\Gamma K$ is a BI of M.
Other cases are also proved in a similar manner.
Corollary3.25:“A Ternary Γsubsemiring D of N is a BI of N if $D=I\Gamma J\Gamma K$, Where I is a right ideal, J is a lateral ideal & K is a left ideal of N.”
Theorem3.26:“Let D be a ternary Γ subsemiring of a TΓSS N. If I is a right ideal, J is a lateral ideal & K is a left ideal of N $\ni I\Gamma J\Gamma K\subseteq D\subseteq I\bigcap J\bigcap K$ then
$D\Gamma N\Gamma D\Gamma N\Gamma D\subseteq(I\bigcap J\bigcap K)\Gamma N\Gamma(I\bigcap J\bigcap K)\Gamma N\Gamma(I\bigcap J\bigcap K)$
$\subseteq I\Gamma(N\Gamma J\Gamma N)\Gamma K\subseteq I\Gamma J\Gamma K\subseteq D$
Thus D is a BI of N. Description of a regular ternarysemiring N in terms of BI & QI of N is as follows”.
Theorem3.27:“In a TΓSS N the following conditions are equivalent.
(i) N is regular
(ii) For every BI B of N, $B\Gamma N\Gamma B\Gamma N\Gamma B=B$
(iii) For every QI W of N, $W\Gamma N\Gamma W\Gamma N\Gamma W=W$."
Proof: (i) $\Rightarrow$(ii)
Suppose N is regular. Let B be a BI of N.Let $b\in B\exists y\in N\ni b=b\alpha y\beta b$.

$b = b\alpha y \beta b\gamma y \delta b \in B\Gamma N\Gamma B\Gamma N\Gamma B$

Thus we find that $B \subseteq B\Gamma N\Gamma B\Gamma N\Gamma B$

$\because$ B is a BI of N, $B\Gamma N\Gamma B\Gamma N\Gamma B \subseteq B$

Consequently we have $B = B\Gamma N\Gamma B\Gamma N\Gamma B$.

Clearly (i) $\Rightarrow$ (ii)

(iii) $\Rightarrow$ (i)

Suppose (iii) holds. Let I be a right ideal, J a lateral ideal & K a left ideal of N.

Then $W = I \cap J \cap K$ is a QI of N, by theorem 3.8, by hypothesis, $W\Gamma N\Gamma W\Gamma N\Gamma W = W$

Now $I \cap J \cap K = W = W\Gamma N\Gamma W\Gamma N\Gamma W \subseteq I\Gamma N\Gamma J\Gamma N\Gamma K \subseteq I\Gamma J\Gamma K$

Again clearly $I\Gamma J\Gamma K \subseteq I \cap J \cap K$ so $I\Gamma J\Gamma K = I \cap J \cap K$

And hence by known Th, N is a regular TΓSS.

Theorem3.28: A Ternary subsemiring S of a regular TΓSS N is a BI of N $\Leftrightarrow$ $S = S\Gamma N\Gamma S$

Proof: If $S = S\Gamma N\Gamma S$, then S is a BI of N.

On the contrary, suppose that S is a BI of regular TΓSS N. Let $s \in S \exists y \in N \ni s = s\alpha y \beta s$

$\Rightarrow s \in S\Gamma N\Gamma S$ & hence $S \subseteq S\Gamma N\Gamma S$

Again, $S\Gamma N\Gamma S \subseteq S\Gamma N\Gamma S\Gamma N\Gamma S \subseteq S$

$\therefore S \subseteq S\Gamma N\Gamma S$

Theorem3.29:"A Ternary subsemiring D of a regular TΓSS V is a bi-ideal of V iff D is QI of V".

Proof: Let V bea regular TΓSS. If D is QI of Vthenit follows that D is a BI of V.

Conversely, Let D be a bi-ideal of V. By known theorem, we find that if Vis a regular TΓSS, then $I \cap J \cap K = I\Gamma J\Gamma K$ for any right ideal I, any lateral ideal J & any left ideal K.

Now $D\Gamma V\Gamma V \cap (V\Gamma D\Gamma V + V\Gamma V\Gamma D\Gamma V\Gamma V) \cap V\Gamma V\Gamma D$

$= D\Gamma V\Gamma V\Gamma (V\Gamma D\Gamma V + V\Gamma V\Gamma D\Gamma V\Gamma V)\Gamma V\Gamma V\Gamma D$

$\subseteq D\Gamma (V\Gamma V\Gamma V)\Gamma D\Gamma (V\Gamma V\Gamma V)\Gamma D + D\Gamma (V\Gamma V\Gamma V)\Gamma V\Gamma D\Gamma (V\Gamma V\Gamma V)\Gamma V\Gamma D$

$\subseteq D\Gamma V\Gamma D\Gamma V\Gamma D + D\Gamma V\Gamma V\Gamma D\Gamma V\Gamma V\Gamma D$

$\subseteq D + D\Gamma V\Gamma D \subseteq D$

Consequently, D is a QI of V.

From above theorem (3.29)& followed by lemma (3.18)wehave the following result.

Theorem3.30: If W_1 & W_2 are two ternaysubsemirings& W_3 is a BI of a regular TΓSS N then $W_1\Gamma W_2\Gamma W_3$, $W_1\Gamma W_3\Gamma W_2$ & $W_3\Gamma W_1\Gamma W_2$ are QIs of N.

From corollary [3.20] &Th(3.30)the result as follows

Corollary3.31: For any three QIs W_1, W_2, W_3 of a regular TΓSS N, $W_1\Gamma W_2\Gamma W_3$ is QI of N.

Conclusion:

"In this paper we introduce the notions of quasi ideal, bi-ideals of Ternary Γ-SO semirings and obtain characterizations of quasi ideals, minimal quasi ideals and bi-ideals in Ternary Γ-SO semiring".

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