Common fixed points and Coincidence points theorems for expansive mapping in b-Metric Spaces

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Abstract:

In this article, we establish common fixed point and coincidence point result for expansive type mappings in b-metric spaces. Overall results extended and generalize several well known comparable results in the literature. Some examples are also presented for the validity of this research paper.

Keyword: Fixed point**;** b - metric spaces, Common fixed point and Coincidence point.

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1. Introduction:

The fixed point theory is beautiful subject in Topology as well as functional analysis. Regarding the feasibility of application of it to the various disciplines, a number of authors have contributed to this theory with a number of publications. One of the simplest and most useful results in fixed point theory is the Banach fixed point theorem [3] : Let (X,d) be a complete metric space and T be self mapping of X satisfying

$$d(Tx, Ty) \leq \lambda d(x,y) \text{ for all } x, y \in X. \tag{1.1}$$

where $\lambda \in [0, 1)$, then T has a unique fixed point. A mapping satisfying the condition (1.1) is called contraction mapping. As well as, there are a lot of extensions of this famous fixed point theorem in metric space which are obtained generalizing contractive condition, there are a lot of generalizations of it in different space which has metric type structure. In fact, Banach demonstrated how to find the desired fixed point by offering a smart and plain technique.

The study of expansive mapping is a very interesting research area in fixed point theory and Mohanta [13]. A mapping satisfying the condition

$$d(Tx, Ty) \geq \lambda d(x, y) \text{ for all } x, y \in X \text{ where } \lambda > 1 \tag{1.2}$$

is called expansive mapping.

The purpose of this work is found sufficient condition of existence of point of common and coincidence fixed points for a pair of self mapping satisfying some expansive type condition in b-metric space. The concept of metric space one such generalization is a b-metric space introduced and studied by Bakhtin [5] and Czerwik [14] contraction mapping in b-metric spaces as a generalization of metric spaces many mathematicians worked on this interesting space for more the reader can refer [2, 6, 10, 11, 12, 15] formally defined a b-metric space in this paper we present common and coincidence fixed point theorem under expansive mapping in b-metric space these result in proves and generalization some important known result in the literature.

2. Preliminaries & Definition:

Theorem 2.1 [1]

Let (X,d) be a b-metric space and let $x_n \to x$ and $y_n \to y$ such that x, y converge to point belong X. Then, we have

$$\tfrac{1}{2} d(x, y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x, y)$$

In particular, if x = y, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Moreover, for each $z \in X$, we have

$$\tfrac{1}{-} d(x, z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x, z).$$

Definition 2.2 [14]

Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is called a b-metric on X if the following properties hold:

(i) $d(x,y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x)$ for all $x, y, \in X$

(iii) $d(x, y) \leq s(d(x,z) + d(z, y))$ for all $x, y, z \in X$.

The pair (X,d) is called a b-metric space.

See that if $s = 1$, then the ordinary triangle inequality in a metric space is satisfied, however it does not hold true when $s > 1$. Thus the class of b-metric spaces is effectively larger than that of the ordinary metric spaces. That is, every metric space is a b-metric space, but the converse need not be true. The following example illustrates the above remarks.

Example 2.3

Let $X = \{-3, 0, 2\}$. Define $d: X \times X \to R^+$ by $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(x, x) = 0$, $x \in X$ and $d(-3, 0) = 5$, $d(-3, 2) = d(0, 2) = 2$. Then (X, d) is a b-metric space, but not a metric space since the triangle inequality is not satisfied. Indeed we have that

$$d(-3, 2) + d(2, 0) = 2 + 2 = 4 < 5 = d(-3, 0)$$

It is easy to verify that $s = \frac{5}{4}$.

Example 2.4 [7]

Let (X, d) be a metric space and $\rho(x, y) = (d(x,y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 3^{p-1}$.

Definition 2.5 [9]

Let (X,d) be a metric space $x \in X$ and (x_n) be a sequence in X. Then

(i) (x_n) converges to x if and only if $\lim_{n\to\infty} d(x_n, x) = 0$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) (x_n) is Cauchy if and only if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

(iii) (X, d) is complete if and only if every Cauchy sequence in X is convergent.

Definition 2.6

Let (X,d) be a b-metric space with the coefficient $s \geq 1$ and let $R : X \to X$ be a given mapping. We say that R is continuous at $x_0 \in X$ if for every sequence (x_n) in X, we have $x_n \to x_0$ as $n \to \infty$ then $Rx_n \to Rx_0$ as $n \to \infty$. If R is continuous at each point $x_0 \in X$, then we say that R is continuous on X.

Definition 2.7

Let (X,d) be a b-metric space with the coefficient $s \geq 1$. A mapping $R : X \to X$ is called expansive if there exists a real constant $k > s$ such that

$$d(Rx, Ry) \geq kd(x, y)$$

for all x, y, $\in$ X.

Definition 2.8 [8]

Let R and S be self mappings of a set X. If $y = Rx = Sx$ for some x in X, then x is called a coincidence point of R and S and y is called a point of coincidence of R and S.

Definition 2.9 [4]

The mappings R, S: $X \to X$ are weakly compatible, if for every $x \in X$, the following holds:

$$R(Sx) = S(Rx) \text{ whenever } Sx = Rx.$$

Definition 2.10

Let S and T be weakly compatible self maps of a nonempty set X. If S and R have a unique point of coincidence $y = Sx = Rx$, then y is the unique common fixed point of S and R.

3. Main Results:

Theorem 3.1

Let (X,d) be a b-metric space with the coefficient $s \geq 1$. Suppose the two self mappings $T: X \to X$ and $g : X \to X$ satisfy the condition.

$$d(Tx, Ty) \geq \alpha d(gx, Tx) + \beta d(gy, Ty) + \gamma d(gx, gy) + \delta d(gx, Ty) \tag{3.1}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma, \delta$ are nonnegative real numbers with $\alpha + \beta + \gamma + \delta > s$.

Assume the following hypotheses.

(i) $\alpha < 1$ and $\gamma \neq 0$,

(ii) $\beta < 1$ and $\delta \neq 0$,

(iii) $g(X) \subseteq T(X)$,

(iv) T(X) or g(X) is complete.

Then T and g have a point of coincidence in X. Moreover, if $\gamma > 1$, then the point of coincidence is unique. If T and g are weakly compatible and $\gamma > 1$, then T and g have a unique common fixed point in X.

Proof:

Let $x_0 \in X$ and choose $x_1 \in X$ such that $gx_0 = Tx_1$. This is possible since $g(X) \subseteq T(X)$. Continuing this process, we can construct a sequence (x_n) in X such that $Tx_n = gx_{n-1}$, for all $n \geq 1$.

By (3.1), we have

$$
\begin{aligned}
d(gx_{n-1}, gx_n) &= d(Tx_n, Tx_{n+1}) \\
&\geq \alpha d(gx_n, Tx_n) + \beta d(gx_{n+1}, Tx_{n+1}) + \gamma d(gx_n, gx_{n+1}) + \delta d(gx_n, Tx_{n+1}) \\
&= \alpha d(gx_n, gx_{n-1}) + \beta d(gx_{n+1}, gx_n) + \gamma d(gx_n, gx_{n+1}) + \delta d(gx_n, gx_n) \\
&= \alpha d(gx_n, gx_{n-1}) + \beta d(gx_{n+1}, gx_n) + \gamma d(gx_n, gx_{n+1})
\end{aligned}
$$

$$(1-\alpha)d(gx_n, gx_{n-1}) \geq (\beta+\gamma)d(gx_n, gx_{n+1})$$

$$\left(\frac{1-\alpha}{\beta+\gamma}\right) d(gx_n, gx_{n-1}) \geq d(gx_n, gx_{n+1})$$

which gives that

$$d(gx_n, gx_{n+1}) \leq \lambda d(gx_{n-1}, gx_n)$$

where $\lambda = \left(\frac{1-\alpha}{\beta+\gamma}\right)$. It is easy to see that $\lambda \in (0, 1/s)$.

By induction, we get that

$$d(gx_n, gx_{n+1}) \leq \lambda^n d(gx_0, gx_1) \tag{3.2}$$

for all $n \geq 0$.

For $m, n \in N$ with $m > n$, we have by repeated use of (3.2)

$$
\begin{aligned}
d(gx_n, gx_m) &\leq s[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_m)] \\
&\leq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + \ldots.. \\
&+ s^{m-n-1}[d(gx_{m-2}, gx_{m-1}) + d(gx_{m-1}, gx_m)] \\
&\leq [s\lambda^n + s^2\lambda^{n+1} + .. + s^{m-n-1}\lambda^{m-2} + s^{m-n}\lambda^{m-1}] d(gx_0, gx_1) \\
&= s\lambda^n [1 + s\lambda + (s\lambda)^2 + .. + (s\lambda)^{m-n-2} + (s\lambda)^{m-n-1}] d(gx_0, gx_1)
\end{aligned}
$$

$$\leq \frac{\lambda}{1-\lambda} d(gx_0, gx_1)$$

So (gx_n) is a Cauchy sequence in g(X). Suppose that g(X) is a complete subspace of X. Then there exists $y \in g(X) \subseteq T(X)$ such that $gx_n \to y$ and also $Tx_n \to y$. In case, T(X) is complete, this holds also with $y \in T(X)$. Let $u \in X$ be such that Tu = y.

By (3.1), we have

$$d(gx_{n-1}, Tu) = d(Tx_n, Tu)$$

$$\geq \alpha d(gx_n, Tx_n) + \beta d(gu, Tu) + \gamma d(gx_n, gu) + \delta(gx_n, Tu)$$

$$\geq \gamma d(gx_n, gu)$$

If $\gamma \neq 0$, then

$$d(gx_n, gu) \leq \frac{1}{\gamma} d(gx_{n-1}, Tu)$$

Therefore,

$$d(y, gu) \leq s[d(y, gx_n) + d(gx_n, gu)]$$

$$\leq s[d(y, gx_n) + \frac{1}{\gamma} d(gx_{n-1}, Tu)]$$

$$= s[d(y, gx_n) + \frac{1}{\gamma} d(Tx_n, Tu)].$$

Taking limit as $n \to \infty$, we have d(y, gu) = 0, i.e., gu = y and hence Tu = gu = y. Therefore, y is a point of coincidence of T and g.

Now, we suppose that $\gamma > 1$. Let v be another point of coincidence of T and g. So Tx = gx = v for some $x \in X$. Then

$$d(y, v) = d(Tu, Tx) \geq \alpha d(gu, Tu) + \beta d(gx, Tx) + \gamma d(gu, gx) + \delta d(gu, Tx) = \gamma d(y, v) + \delta d(y, v),$$

which implies that

$$d(y, v) \geq (\gamma + \delta)\, d(y, v)$$

$$d(y, v) \leq \frac{1}{\gamma+\delta} d(y, v).$$

Taking $\gamma + \delta = \in$

Since $\in > 1$, we have $d(v, y) = 0$ i.e., $v = y$.

Therefore, T and g have a unique common point in X.

If T and g are weakly compatible, then by definition 2.10, T and g have a unique common fixed point in X.

Theorem 3.2:

Let (X,d) be a b-metric space with the coefficient $s \geq 1$. Suppose the mapping T, g, : $X \rightarrow X$ satisfy the condition.

$$d(Tx, Ty) \geq \gamma(gx, gy) + \delta(gx, Ty)$$

for all x, y, $\in$ X, where $\gamma > s$ is a constant. If $g(X) \subseteq T(X)$ and T(X) or g(X) is complete, then T and g have a unique point of coincidence in X, moreover, if T and g are weakly compatible, then T and g have a unique common fixed point in X.

Proof: It follows by taking $\beta = \alpha = 0$ in Theorem 3.1

Theorem 3.3 Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$. Suppose the mappings S, R : $X \rightarrow X$ satisfy the following conditions:

$$d(R(Sx), Sx) + -d(R(Sx), x) \geq \gamma d(Sx, x) \quad (3.3)$$

and

$$d(S(Rx), Rx) + -d(S(Rx), x) \geq \alpha d(Rx, x) \quad (3.4)$$

for all $x \in X$, where γ, β, k are nonnegative real numbers with $\gamma > s + (1 + s)k$ and $\alpha > s + (1 + s) k$. If S and R are continuous and surjective, then S and R have a common fixed point in X.

Proof: Let $x_0 \in X$ be arbitrary and choose $x_1 \in X$ such that $x_0 = Rx_1$. This is possible since T is surjective. Since S is also surjective, there exists $x_2 \in X$ such that $x_1 = Sx_2$. Contiuting this process, we can construct a sequence (x_n) in X such that $x_{2n} = Rx_{2n+1}$ and $x_{2n-1} = sx_{2n}$ for all $n \in N$.

Using (3.3), we have for $n \in N \cup \{0\}$

$$d(R(Sx_{2n+2}), Sx_{2n+2}) + d(R(Sx_{2n+2}), x_{2n+2}) \geq \gamma d(Sx_{2n+2}, x_{2n+2})$$

which implies that

$$d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+2}) \geq \gamma d(x_{2n+1}, x_{2n+2}).$$

Hence, we have

$$\gamma d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}) + kd(x_{2n}, x_{2n+1}) + kd(x_{2n+1}, x_{2n+2}).$$

Therefore,

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{1+}{\gamma-} d(x_{2n}, x_{2n+1}). \qquad (3.5)$$

Using (3.4) and by an argument similar to that used above, we obtain that

$$d(x_{2n}, x_{2n+1}) \leq \frac{1+}{\alpha-} d(x_{2n-1}, x_{2n}). \qquad (3.6)$$

Let $\lambda = \max\{\frac{1+}{\gamma-}, \frac{1+}{\alpha-}\}$. It is easy to see that $\lambda \in (0, 1/s)$.

Combining (3.5) and (3.6), we get

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n) \qquad (3.7)$$

for all $n \geq 1$. By repeated application of (3.7), we obtain

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1),$$

By an argument similar to that used in theorem 3.1, it follows that (x_n) is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. Now, $x_{2n+1} \to u$ and $x_{2n} \to u$ as $n \to \infty$. The continuity of S and R imply that $Rx_{2n+1} \to Ru$ and $Sx_{2n} \to Su$ as $n \to \infty$ i.e., $x_{2n} \to Ru$ and $x_{2n-1} \to Su$ as $n \to \infty$. The uniqueness of limit yields that u = su = Ru. Hence, u is a common fixed point of S and R.

Theorem 3.4 Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$. Let $R : X \to X$ be a continuous surjective mapping such that

$$d(R^2x, Rx) + -d(R^2x, x) \geq \gamma d(Rx, x)$$

for all $x \in X$, where γ, k are nonnegative real numbers with $\gamma > s + (1 + s) k$. Then T has a fixed point in X.

Proof: It follows from Theorem (3.3) by taking S = R and $\alpha = \gamma$.

Theorem 3.5 Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$. Let $R : X \to X$ be a continuous surjective mapping such that

$$d(R^2x, Rx) \geq \gamma d(Rx, x)$$

for all $x \in X$, where $\gamma > s$ is a constant. Then has a fixed point in X.

Proof: It follows from Theorem (3.3) by taking S = R and $\alpha = \gamma$, k = 0.

Example: 3.6

Let $X = [0,\infty)$ and define $d: X \times X \to R^+$ by $d(x, y) = |x-y|^2$, $\forall\, x, y \in X$. Then (X,d) is a complete b-metric space with s = 2. Define $R: X \to X$ by R(x) = 3x. Then R has a fixed point.

Proof We have example for Theorem 3.4

$$d(R(Rx), Rx) + - d(Rx), x) = d(9x , 3x) + d(9x, x)$$

$$= |9x - 3x|^2 + |9x - x|^2$$

$$= 36x^2 + 64x^2$$

$= 100x^2$

$= 25\ |3x - x|^2$

$= 25\ d(Rx, x)$

for all $x \in X$. Here $k = 2$ and $\gamma = 25$. Clearly $25 = \gamma > s + (1 + s)k$

$= 2 + (1 + 2)\ k$

$= 2 + 3k$

$= 2 + 3.2$

$= 2 + 6$

$= 8$

Also R is surjectioin on X. Thus R satisfies all the hypotheses of theorem 3.4 and hence R has a fixed point. Here $0 \in X$ is the fixed point of R.

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