Fixed Point Theorems for single-valued and multi-valued operator on spaces with vector-valued b-matrices

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Abstract:

The aim of this paper is to find the some fixed point theorems for generalized single-valued and multi-valued contractive on a set endowed with one or two vector-valued b-matrices.

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1. Introduction:

The Fixed point theory is well-known in the current literature because it provides useful tools for solving many problems with applications in the different fields such as computer science, engineering, chemistry, game theory and economics. There has been a number of generalizations of the usual notion of the metric space one such generalization is a b-metric space introduced and studied by Bakhtin [1] and Czerwik [4] since then various papers gives with fixed point theory for single-valued and multi-valued operators in b-metric spaces [2] [4] [9]. The aim of this paper is to find fixed point theorems for generalized single-valued and multi-valued contractions on generalized b-metric space. In this paper we present some results on generalized b-metric space and two generalized b-metric space for two generalized b-metric space using results [6], [7], [8], [9].

In this paper proved some unique fixed point results for an operator T-satisfying contractive conditions in complete generalized b-metric space for this paper used the results from [3].

2. Preliminaries and Definitions:

Definition 2.1

Let X be a non-empty set and A function d : X × X → R is called a metric on X if and only if the following properties holds.

1. d(x, y) = 0 if and only if x = y;
2. d(x, y) = d(y, x) for all x, y ∈ X;
3. d(x, y) ≤ d(x, z) + d(z, y) for all x, y, z ∈ X.

The number d(x, y) is called the distance between x and y and the pair (X, d) is called a metric space on X.

Example 2.1

Let E_n(or R^n) = {x : (x_1, x_2,...., x_n), x_i ∈ R, R the set of real numbers} and let d be defined as follows : if y = (y_1, y_2,...., y_n) then

$$d(x, y) = \left(\sum_i^n |x_i - y_i|^p\right)^{\frac{1}{p}} = dp(x, y);$$

where p is a fixed number in [1, ∞)

Definition 2.2[7]

Let X be a non-empty set and s ≥ 1 be a given real number. A functions d : X × X → R^+ is said to be a b-metric on X if the following conditions hold :

1. d(x, y) = 0 if and only if x = y;
2. d(x, y) = d(y, x) for all x, y ∈ X;
3. d(x, y) ≤ s(d(x, z) + d(z, y)) for all x, y, z ∈ X.

The pair (X, d) is called b-metric space we see that if s = 1, then the ordinary triangle inequality in a metric space is satisfied.

However, if does not hold true when s > 1 that is every metric space is a b-metric space but the convex need not be true. The following example illustrates the above remarks.

Example 2.2

Let X = {-1, 0, 1}. Define

d : X × X → R^+ by d (x, y) = d(y, x) for all x, y ∈ X, d(x, x) = 0, x ∈X and d(-1, 0) = 3, d(-1, 1) = d(0, 1) = 1

Then (X, d) is a b-metric space but not a metric space since the triangle inequality is not hold. Indeed, we have

d(-1, 0) ≤ d(-1, 1) + d(1, 0)

3 ≤ 1 + 1 = 2

But if easy to verify that s = $\frac{3}{2}$

Definition 2.3

Let X be a non-empty set and let s ≥ 1 be a given real number. A function d : X × X → R_+^n is said to be a vector-valued b-metric on X if the following conditions hold :

1. d(x, y) = 0 if and only if x = y;

2. d(x, y) = d(y, x) for all x, y ∈ X;

3. d(x, y) ≤ s(d(x, z) + d(z, y)) for all x, y, z ∈ X.

The pair (X, d) is called a generalized b-metric space

Remark (a) if n = 1, s = 1 we get the the b-metric on X.

(b) if n = 1, we get the b-metric on X introduced by Bakhtin.

Remark if α, β ∈ R^n with α = (α_1,...., α_n) and β = (β_1, ..., β_n) and c ∈ R, by α ≤ β we mean $\alpha_i \leq \beta_i$ for all i ∈ N* and by α ≤ c we mean $\alpha_i \leq c_i$ for all i ∈ N*.

Definition 2.4

Let (X, d) be a generalized b-metric space x ∈ X and (x_n) be a sequence in X then

(a) (x_n) converges if and only if there exits $x \in X$ such that for all $\in > 0$ there exists $n(\in) \in N$ such that for all $n \geq n(\in)$ we have $d(x_n, x) < \in$, In this case we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$

(b) (x_n) is Cauchy if and only if for all $\in > 0$ there exists $n(\in) \in N$ such that for each $n, m \geq n(\in)$, we have

$d(x_n, x_m) < \in$ or $\lim_{n,m\to\infty} d(x_n, x_m) = 0$

(c) (X, d) is complete if and only if every Cauchy sequence in X is convergent.

Definition 2.5

Let (X, d) be a generalized b-metric space, then a subset $Y \subset X$ is called

(i) Y is compact if and only if for every sequence of elements of Y there exists a subsequence that converges to an element of Y.

(ii) Y is closed if and only if for each sequence $(x_n)_{n\in N}$ in Y which converges to an element x, we have $x \in Y$.

Definition 2.6

A metric $C \in M_{n\times n}(R_+)$ is said to be convergent to zero if

$C^k \to 0$ as $k \to \infty$

Lemma 2.1 [8] A metric $C \in M_{n\times n}(R_+)$ is convergent to zero if and only if I - C is nonsingular and $(I - C)^{-1} = I + C + C^2 + ...$

Lemma 2.2 Let (X, d) be a generalized b-metric space and Let $\{x_k\}_{k=0}^{n} \subset X$ Then $d(x_n, x_0) \leq sd(x_0, x_1) + \quad + s^{n-1}d(x_{n-2}, x_{n-1}) + s^n d(x_{n-1}, x_n)$.

Main Results:

Theorem 3.1:

Let (X, d) be a complete generalized b-metric space and $T: X \to X$ be single valued operator satisfies the following conditions:

(a) T is continuous;

(b) There exists matrices $\alpha, \beta, \gamma, \delta \in M_{n\times n}(R_+)$ with

(i) $(I - \alpha - \gamma - \delta_s)$ is nonsingular and $(I - \alpha - \gamma - \delta_s)^{-1} \in M_{n\times n}(R_+)$;

(ii) sC is convergent to zero, we have

$$C = (I - \alpha - \gamma - \delta_s)^{-1}(\alpha + \beta + \delta_s);$$

(iii) $d(Tx, Ty) \leq \alpha\{d(x, Tx) + d(y, Ty)\} + \beta\{d(x, Tx) + d(y, Tx)\}$

$+ \gamma\{d(y, Ty) + d(y, Tx)\} + \delta\{d(x, Ty) + d(y, Tx)\}$

then prove that

1. X* is fixed point of T

2. If, in addition, $(I - \beta - \gamma - 2\delta)$ is nonsingular and $(I - \beta - \gamma - 2\delta)^{-1} \in M_{n\times n}(R_+)$

Then X* is unique fixed point of T.

Proof:

(i) Let $x_0 \in X$. Consider the sequence of successive approximations for T starting from x_0 i.e. $x_{n+1} = Tx_n$, we have

$$d(x_1, x_2) = d(Tx_0, Tx_1)$$

$$\leq \alpha\{d(x_0, Tx_0) + d(x_1, Tx_1)\} + \beta\{d(x_0, Tx_0) + d(x_1, Tx_0)\}$$

$$+ \gamma\{d(x_1, Tx_1) + d(x_1, Tx_0)\} + \delta\{d(x_0, Tx_1) + d(x_1, Tx_0)\}$$

$$\leq \alpha\{d(x_0, x_1) + d(x_1, x_2)\} + \beta\{d(x_0, x_1) + d(x_1, x_1)\}$$

$$+ \gamma\{d(x_1, x_2) + d(x_1, x_2)\} + \delta\{d(x_0, x_2) + d(x_1, x_2)\}$$

$$= \alpha d(x_0, x_1) + \alpha d(x_1, x_2) + \beta d(x_0, x_1) + \gamma d(x_1, x_2) + \delta d(x_0, x_2)$$

$$\leq \alpha d(x_0, x_1) + \alpha d(x_1, x_2) + \beta d(x_0, x_1)$$

$$+ \gamma d(x_1, x_2) + \delta_s d(x_0, x_1) + \delta_s d(x_1, x_2)$$

$d(x_1, x_2) - \alpha d(x_1, x_2) - \gamma d(x_1, x_2) - \delta_s d(x_1, x_2)$

$\leq \alpha d(x_0, x_1) + \beta d(x_0, x_1) + \delta_s(x_0, x_1)$

$(I - \alpha - \gamma - \delta_s)\, d(x_1, x_2) \leq (\alpha + \beta + \delta_s)\, d(x_0, x_1)$

$\Rightarrow \quad d(x_1, x_2) \leq (I - \alpha - \gamma - \delta_s)^{-1}\, (\alpha + \beta + \delta_s)\, d(x_0, x_1)$

$\Rightarrow \quad d(x_1, x_2) \leq Cd(x_0, x_1)$

For Next, we have

$d(x_2, x_3) = (Tx_1, Tx_2)$

$\leq \alpha\{d(x_1, Tx_1) + d(x_2, Tx_2)\} + \beta\{d(x_1, Tx_2) + d(x_2, Tx_1)\}$

$+ \gamma\{d(x_2, Tx_2) + d(x_2, Tx_1)\} + \delta\{d(x_1, Tx_2) + d(x_2, Tx_1)\}$

$\leq \alpha\{d(x_1, x_1) + d(x_2, x_3)\} + \beta\{d(x_1, x_2) + d(x_2, x_2)\}$

$+ \gamma\{d(x_2, x_3) + d(x_2, x_2)\} + \delta\{d(x_1, x_3) + d(x_2, x_2)\}$

$= \alpha d(x_1, x_2) + \alpha d(x_2, x_3) + \beta d(x_1, x_2) + \gamma d(x_2, x_3) + \delta d(x_1, x_3)$

$\leq \alpha d(x_1, x_2) + \alpha d(x_2, x_3) + \beta d(x_1, x_2)$

$+ \gamma d(x_2, x_3) + \delta_s d(x_1, x_2) + \delta_s d(x_2, x_3)$

$d(x_2, x_3) - \alpha d(x_2, x_3) - \gamma d(x_2, x_3) - \delta_s d(x_2, x_3)$

$\leq \alpha d(x_1, x_2) + \beta d(x_1, x_2) + \delta_s d(x_1, x_2)$

$(I - \alpha - \gamma - \delta_s)\, d(x_2, x_3) \leq (\alpha + \beta + \delta_s)\, d(x_1, x_2)$

$d(x_2, x_3) \leq (I - \alpha - \gamma - \delta_s)^{-1}\, (\alpha + \beta + \delta_s)\, d(x_1, x_2)$

$d(x_2, x_3) \leq Cd(x_1, x_2)$

$d(x_2, x_3) \leq C\{Cd(x_0, x_1)\}$

$d(x_2, x_3) \leq C^2 d(x_0, x_1)$

Again

$d(x_3, x_4) = d(Tx_2, Tx_3)$

$\leq \alpha\{d(x_2, x_3) + \alpha d(x_3, x_4) + \beta d(x_2, x_3) + \gamma d(x_3, x_4) + \delta_s d(x_2, x_3) + \delta_s d(x_3, x_4)$

$d(x_3, x_4) \leq (I - \alpha - \gamma - \delta_s)^{-1} (\alpha + \beta + \delta_s) d(x_2, x_3)$

$d(x_3, x_4) \leq Cd (x_2, x_3)$

$d(x_3, x_4) \leq C^3 d (x_0, x_1)$

This implies that

$d(x_3, x_4) \leq C^3 d (x_0, x_1)$

By induction method, we get

$d(x_k, x_{k+1}) \leq C^k d (x_0, x_1), \forall k \in N$

We will show that $(x_k)_{k\in N}$ is Cauchy sequence by estimating $d(x_k, x_{k+p})$

$d(x_k, x_{k+p}) \leq sd(x_k, x_{k+1}) + s^2 d(x_{k+1}, x_{k+2}) + s^3 d (x_{k+2}, x_{k+3})$

$+...+ s^{p-2} d(x_{k+p-3}, x_{k+p-2}) + s^{p-1} d(x_{k+p-2}, x_{k+p-1}) + s^{p-1} d(x_{k+p-1}, x_{k+p})$

$d(x_k, x_{k+p}) \leq sC^k d(x_0, x_1) + s^2 C^{k+1} d(x_0, x_1) + s^3 C^{k+2} d (x_0, x_1)$

$+...+ s^{p-2} C^{k+p-3} d(x_0, x_1) + s^{p-1} C^{k+p-2} d(x_0, x_1) + s^{p-1} C^{k+p-1} d(x_0, x_1)$

$d(x_k, x_{k+p}) \leq sC^k d(x_0, x_1) [I + sC + s^2C^2 + ... + s^{p-2} C^{p-2} + s^{p-2} C^{p-1}]$

$d(x_k, x_{k+p}) \leq sC^k d(x_0, x_1) [I + sC + s^2C^2 + ... + s^{p-2} C^{p-2} + s^{p+} C^{p+}]$

$d(x_k, x_{k+p}) \leq sC^k d(x_0, x_1) (I - sC)^{-1} \leq (sC)^k d(x_0, x_1)(I - sC)^{-1}$

I-sC is nonsingular since sC is convergent to zero,

$\Rightarrow$ The sequence $(x_k)_{k\in N}$ is Cauchy sequence

But (X, d) is complete

$\Rightarrow \exists x^* \in X$ such that $d(x_k, x^*) \rightarrow 0$ as $k\rightarrow\infty$ (1)

By (a) we have

$d(Tx_{k-1}, Tx^*) \rightarrow 0$ as $k\rightarrow\infty$ (2)

But $d(Tx_{k-1}, Tx^*) = d(x_k , Tx^*)$ (3)

Hence from (1) and (3), we have

$x^* = Tx^*$

So x^* is a fixed point of T

2. for uniqueness of fixed point

Let x^* and y be two fixed point of T, then $Tx^* = x^*$ and $Ty = y$ we have

$d(x^*, y) = (Tx^*, Ty)$

$\leq \alpha\{d(x^*, Tx^*) + d(y, Ty)\} + \beta\{d(x^*, Tx^*) + d(y, Tx^*)\}$

$+ \gamma\{d(y, Ty) + d(y, Tx^*)\} + \delta\{d(x^*, Ty) + d(y, Tx^*)\}$

$\leq \alpha d(x^*, x^*) + \alpha d(y, y) + \beta d(x^*, x^*) + \beta d(y, x^*)$

$+ \gamma\, d(y, y) + \gamma d(y, x^*) + \delta d(x^*, y) + \delta d(y, x^*)$

$d(x^*, y) \leq \beta d(y, x^*) + \gamma d(y, x^*) - 2\delta d(x^*, y)$

$(I - \beta - \gamma- 2\delta)\, d(x^*, y) \leq 0$

$\Rightarrow$ $x^* = y$

Hence x^* is the unique fixed point.

Theorem 3.2

Let (X, d) be a complete generalized b-metric space and $\phi : X \to pd(X)$ be a multivalued operator such that there exists matrices $\alpha, \beta, \gamma, \delta \in M_{n\times n}(R_+)$ with

(i) $(I - \alpha - \gamma - \delta_s)$ is nonsingular and $(I - \alpha - \gamma - \delta_s)^{-1} \in M_{n\times n} (R_+)$

(ii) $(I - \delta_s)$ is nonsingular and $(I - \delta_s)^{-1} \in M_{n\times n}(R_+)$

(iii) $[I - s(I - \delta_s)^{-1} (\alpha + \gamma)]$ is nonsingular and $[I - s(I - \delta_s)^{-1} (\alpha + \gamma)] \in M_{n\times n}(R_+)$

(iv) sC is convergent to zero, where

$C = (I - \alpha - \gamma - \delta_s)^{-1} (\alpha + \beta + \delta_s)$

(v) for each x, y, ∈ X and each $u \in \phi(n)$ there exists $\upsilon \in \phi(y)$ such that

$$d(u, \upsilon) \leq \alpha\{d(x, u) + d(y, \upsilon)\} + \beta\{d(x, u) + d(y, u)\}$$

$$+ \gamma\{d(y, \upsilon) + d(y, u)\} + \delta\{d(x, \upsilon) + d(y, u)\}$$

Then ϕ has a fixed point x* in X i.e. $x^* \in \phi(x^*)$

Proof:

Let $x_0 \in X$ and $x_1 \in \phi(x_0)$, there exists $x_2 \in \phi(x_1)$ such that

$$d(x_1, x_2) \leq \alpha\{d(x_0, x_1) + d(x_1, x_2)\} + \beta\{d(x_0, x_1) + d(x_1, x_1)\}$$

$$+ \gamma\{d(x_1, x_2) + d(x_1, x_1)\} + \delta\{d(x_0, x_2) + d(x_1, x_1)\}$$

$$d(x_1, x_2) \leq \alpha d(x_0, x_1) + \alpha d(x_1, x_2) + \beta d(x_0, x_1) + \gamma d(x_1, x_2) + \delta d(x_0, x_2)$$

$$d(x_1, x_2) \leq \alpha d(x_0, x_1) + \alpha d(x_1, x_2) + \beta d(x_0, x_1) + \gamma d(x_1, x_2) + \delta_s d(x_0, x_1) + \delta_s d(x_1, x_2)$$

$$d(x_1, x_2) - \alpha d(x_1, x_2) - \gamma d(x_1, x_2) - \delta_s d(x_1,x_2) \leq \alpha d(x_0, x_1) + \beta d(x_0, x_1) + \delta_s d(x_0, x_1)$$

$$(I - \alpha - \gamma - \delta_s)\, d(x_1, x_2) \leq (\alpha + \beta + \delta_s)\, d(x_0, x_1)$$

$$d(x_1, x_2) \leq (I - \alpha - \gamma - \delta_s)^{-1} (\alpha + \beta + \delta_s)\, d(x_0, x_1)$$

$$d(x_1, x_2) \leq C d(x_0, x_1)$$

Now $x_2 \in \phi(x_1)$, there exists $x_3 \in \phi(x_2)$ such that

$$d(x_2, x_3) \leq \alpha\{d(x_1, x_2) + d(x_2, x_3)\} + \beta\{d(x_1, x_2) + d(x_2, x_2)\}$$

$$+ \gamma\{d(x_2, x_3) + d(x_2, x_2)\} + \delta\{d(x_1, x_3) + d(x_2, x_2)\}$$

$$d(x_2, x_3) \leq \alpha d(x_1, x_2) + \alpha d(x_2, x_3) + \beta d(x_1, x_2) + \gamma d(x_2, x_3) + \delta d(x_1, x_3)$$

$$d(x_2, x_3) \leq \alpha d(x_1, x_2) + \alpha d(x_2, x_3) + \beta d(x_1, x_2) + \gamma d(x_2, x_3) + \delta_s d(x_1, x_2) + \delta_s d(x_2, x_3)$$

$$d(x_2, x_3) - \alpha d(x_2, x_3) - \gamma d(x_2, x_3) - \delta_s d(x_2, x_3) \leq \alpha d(x_1, x_2) + \beta d(x_1, x_2) + \delta_s d(x_1, x_2)$$

$$(I - \alpha - \gamma - \delta_s)\, d(x_2, x_3) \leq (\alpha + \beta + \delta_s)\, d(x_1, x_2)$$

$d(x_2, x_3) \leq (I - \alpha - \gamma - \delta_s)^{-1} (\alpha + \beta + \delta_s) d(x_1, x_2)$

$d(x_2, x_3) \leq Cd(x_1, x_2)$

$d(x_2, x_3) \leq C\{Cd(x_0, x_1)\}$

$d(x_2, x_3) \leq C^2 d(x_0, x_1)$

similarly

$d(x_3, x_4) \leq C^3 d(x_0, x_1)$

By an induction method, we can construct a sequence $(x_k)_{k \in N}$ in X such that

$$x_k \in \phi(x_{k-1}),\ k \in N$$

and $\quad d(x_k, x_{k+1}) \leq C^k d(x_0, x_1) \quad k \in N$

We will prove next that $(x_k)_{k \in N}$ is Cauchy, estimating $d(x_k, x_{k+p})$.

Thus we have

$$d(x_k, x_{k+p}) \leq sd(x_k, x_{k+1}) + s^2 d(x_{k+1}, x_{k+2}) + ... + s^{p-2} d\,(x_{k+p-3}, x_{k+p-2})$$

$$+ s^{p-1}\, d(x_{k+p-2}, x_{k+p-1}) + s^{p-1}\, d(x_{k+p-1}, x_{k+p})$$

$$\leq sC^k d(x_0, x_1) + s^2 C^{k+1} d(x_0, x_1) + ... + s^{p-2} C^{k+p-3} d\,(x_0, x_1)$$

$$+ s^{p-1} C^{k+p-2}\, d(x_0, x_1) + s^{p-1}\, C^{k+p-1} d(x_0, x_1)$$

$$\leq sC^k d(x_0, x_1)\,[I + sC - 1\ .... + s^{p-2} C^{p-2} + s^{p-2}\, C^{p-1}]$$

$$\leq sC^k d(x_0, x_1)\,[I + sC + ... + s^{p-2} C^{p-2} + s^{p-1}\, C^{p-1}]$$

$$\leq sC^k d(x_0, x_1)\,(I - sC)^{-1}$$

$(x_k, x_{k+p}) \leq (sC)^k\, d(x_0, x_1)(I - sC)^{-1}$

Note that (I-sC) is nonsingular since sC is convergent to zero

$\Rightarrow$ that the sequence $(x_k)_{k \in N}$ is Cauchy from the fact that (X, d) is complete

We have that there exists $x^* \in X$ such that $d(x_k, x^*) \to 0$ as $k \to \infty$

For $x_k \in \phi(x_{k-1})$ there exists $u_k \in \phi(x^*)$ such that

$$d(x_k, u_k) \leq \alpha\{d(x_{k-1}, x_k) + d(x^*, u_k)\} + \beta\{d(x_{k-1}, x_k) + d(x^*, x_k)\}$$

$$+ \gamma\{d(x^*, u_k) + d(x^*, x_k)\} + \delta\{d(x_{k-1}, u_k) + d(x^*, x_k)\}$$

$$\leq \alpha d(x_{k-1}, x_k) + \alpha d(x^*, u_k) + \beta d(x_{k-1}, x_k) + \beta d(x^*, x_k)$$

$$+ \gamma d(x^*, u_k) + \gamma d(x^*, x_k) + \delta_s d(x_{k-1}, x_k) + \delta_s d(x_k, u_k) + \delta d(x^*, x_k)$$

$$(I - \delta_s)\, d(x_k, u_k) \leq \alpha d(x_{k-1}, x_k) + \alpha d(x^*, u_k) + \beta d(x_{k-1}, x_k) + \beta d(x^*, x_k)$$

$$+ \gamma d(x^*, u_k) + \gamma d(x^*, x_k) + \delta_s d(x_{k-1}, x_k) + \delta d(x^*, x_k)$$

$$d(x_k, u_k) \leq (1 - \delta_s)^{-1} [\alpha d(x_{k-1}, x_k) + \alpha d(x^*, u_k) + \beta d(x_{k-1}, x_k) + \beta(x^*, x_k)$$

$$+ d(x^*, u_k) + \gamma d(x^*, x_k) + \delta_s d(x_{k-1}, x_k) + \delta d(x^*, x_k)]$$

We will next estimate $d(x^*, u_k)$ and obtain

$$d(x^*, u_k) \leq s[d(x^*, x_k) + d(x_k, u_k)]$$

$$d(x^*, u_k) \leq s[d(x^*, x_k) + (I - \delta_s)^{-1}\{\alpha d(x_{k-1}, x_k) + \alpha d(x^*, u_k) + \beta d(x_{k-1}, x_k) + \beta d(x^*, x_k)$$

$$+ \gamma d(x^*, u_k) + \gamma d(x^*, x_k) + \delta_s d(x_{k-1}, x_k) + \delta d(x^*, x_k)\}]$$

$$d(x^*, u_k) \leq s\,(I - \delta_s)^{-1}\alpha d(x^*, u_k) + s(I - \delta_s)^{-1}\gamma d(x^*, u_k) + s\{d(x^*, x_k) + (I - \delta_s)^{-1}$$

$$[\alpha d(x_{k-1}, x_k) + \beta d(x_{k-1}, x_k) + \beta d(x^*, x_k) + \gamma d(x^*, x_k) + \delta_s d(x_{k-1}, x_k) + \delta d(x^*, x_k)]\}$$

$$[I - s(I - \delta_s)^{-1}(\alpha + \gamma)]\, d(x^*, u_k) \leq s\{d(x^*, x_k) + (I - \delta_s)^{-1} [\alpha d(x_{k-1}, x_k) + \beta d(x_{k-1}, x_k)$$

$$+ \beta d(x^*, x_k) + \gamma d(x^*, x_k) + \delta_s d(x_{k-1}, x_k) + \delta d(x^*, x_k)]\} \to 0 \text{ as } k \to \infty$$

Hence $u_k \to x^*$

But $u_k \in \phi(x^*)$ and $\phi(x^*)$ is closed

So we get $x^* \in \phi(x^*)$.

Theorem 3.3

Let (X, d_1) be a complete generalized b-metric space and d another vector-valued b-metric on X and T: $X \to X$ be single operator satisfies the following conditions:

(a) There exists a matrix $U \in M_{n\times n}(R_+)$ such that $d_1(x, y) \leq U \,.\, d(x, y)$, for all $x, y \in X$;

(b) T is (d_1, d_1) continuous;

(c) There exists matrices $\alpha, \beta, \gamma, \delta \in M_{n\times n}(R_+)$ with :

(i) $(I - \alpha - \gamma - \delta_s)$ is nonsingular and $(I - \alpha - \gamma - \delta_s)^{-1} \in M_{n\times n}(R_+)$;

(ii) sC is convergent to zero, we have

$$C = (I - \alpha - \gamma - \delta_s)^{-1} (\alpha + \beta + \delta_s);$$

(iii) $d(Tx, Ty) \leq \alpha\,\{d(x, Tx) + d(y, Ty)\} + \beta\{d(x, Tx) + d(y, Tx)\}$

$$+ \gamma\{d(y, Ty) + d(y, Tx)\} + \delta\{d(x, Ty) + d(y, Tx)\}$$

then prove that

1. For any $x_0 \in X$, we have

   $d_1(T^k x_0, x^*) \to 0$ as $k \to \infty$, where x^* is a fixed point of T

2. If, addition, $(I - \beta - \gamma - 2\delta)$ is nonsingular and $(I - \beta - \gamma - 2\delta)^{-1} \in M_{n\times n}(R_+)$

   Then X^* is unique fixed point of T.

Proof:

(1) Let $x_0 \in X$. consider the sequence of successive approximations for T starting from x_0 i.e. $x_{n+1} = Tx_n$, we have

$d(x_1, x_2) = d(Tx_0, Tx_1)$

$$\leq \alpha\{d(x_0, Tx_0) + d(x_1, Tx_1)\} + \beta\{d(x_0, Tx_0) + d(x_1, Tx_0)\}$$

$+ \gamma\{d(x_1, Tx_1) + d(x_1, Tx_0)\} + \delta\{d(x_0, Tx_1) + d(x_1, Tx_0)\}$

$\leq \alpha\{d(x_0, x_1) + d(x_1, x_2)\} + \beta\{d(x_0, x_1) + d(x_1, x_1)\}$

$+ \gamma\{d(x_1, x_2) + d(x_1, x_2)\} + \delta\{d(x_0, x_2) + d(x_1, x_2)\}$

$= \alpha d(x_0, x_1) + \alpha d(x_1, x_2) + \beta d(x_0, x_1) + \gamma d(x_1, x_2) + \delta d(x_0, x_2)$

$\leq \alpha d(x_0, x_1) + \alpha d(x_1, x_2) + \beta d(x_0, x_1) + \gamma d(x_1, x_2) + \delta_s d(x_0, x_1)$

$+ \delta_s d(x_1, x_2)$

$d(x_1, x_2) - \alpha d(x_1, x_2) - \gamma d(x_1, x_2) - \delta_s d(x_1, x_2)$

$\leq \alpha d(x_0, x_1) + \beta d(x_0, x_1) + \delta_s(x_0, x_1)$

$(I - \alpha - \gamma - \delta_s)\, d(x_1, x_2) \leq (\alpha + \beta + \delta_s)\, d(x_0, x_1)$

$\Rightarrow \quad d(x_1, x_2) \leq (I - \alpha - \gamma - \delta_s)^{-1} (\alpha + \beta + \delta_s)\, d(x_0, x_1)$

$\Rightarrow \quad d(x_1, x_2) \leq Cd(x_0, x_1)$

For Next, we have

$d(x_2, x_3) = (Tx_1, Tx_2)$

$\leq \alpha\{d(x_1, Tx_1) + d(x_2, Tx_2)\} + \beta\{d(x_1, Tx_2) + d(x_2, Tx_1)\}$

$+ \gamma\{d(x_2, Tx_2) + d(x_2, Tx_1)\} + \delta\{d(x_1, Tx_2) + d(x_2, Tx_1)\}$

$\leq \alpha\{d(x_1, x_1) + d(x_2, x_3)\} + \beta\{d(x_1, x_2) + d(x_2, x_2)\}$

$+ \gamma\{d(x_2, x_3) + d(x_2, x_2)\} + \delta\{d(x_1, x_3) + d(x_2, x_2)\}$

$= \alpha d(x_1, x_2) + \alpha d(x_2, x_3)\} + \beta d(x_1, x_2) + \gamma d(x_2, x_3) + \delta\{d(x_1, x_3)$

$\leq \alpha d(x_1, x_2) + \alpha d(x_2, x_3)\} + \beta d(x_1, x_2)$

$+ \gamma d(x_2, x_3) + \delta_s d(x_1, x_2) + \delta_s d(x_2, x_3)$

$d(x_2, x_3) - \alpha d(x_2, x_3) - \gamma d(x_2, x_3) - \delta_s d(x_2, x_3)$

$\leq \alpha d(x_1, x_2) + \beta d(x_1, x_2) + \delta_s d(x_1, x_2)$

$(I - \alpha - \gamma - \delta_s)\, d(x_2, x_3) \leq (\alpha + \beta + \delta_s)\, d(x_1, x_2)$

$d(x_2, x_3) \leq (I - \alpha - \gamma - \delta_s)^{-1} (\alpha + \beta + \delta_s)\, d(x_1, x_2)$

$d(x_2, x_3) \leq Cd(x_1, x_2)$

$d(x_2, x_3) \leq C\{Cd(x_0, x_1)\}$

$d(x_2, x_3) \leq C^2 d(x_0, x_1)$

Again

$d(x_3, x_4) = d(Tx_2, Tx_3)$

$\leq \alpha d(x_2, x_3) + \alpha d(x_3, x_4) + \beta d(x_2, x_3) + \gamma d(x_3, x_4) + \delta_s d(x_2, x_3) + \delta_s d(x_3, x_4)$

$d(x_3, x_4) \leq (I - \alpha - \gamma - \delta_s)^{-1} (\alpha + \beta + \delta_s)\, d(x_2, x_3)$

$d(x_3, x_4) \leq Cd\, (x_2, x_3)$

$d(x_3, x_4) \leq C^3 d\, (x_0, x_1)$

This implies that

$d(x_3, x_4) \leq C^3 d\, (x_0, x_1)$

By induction method, we get

$d(x_k, x_{k+1}) \leq C^k d\, (x_0, x_1),\ \forall\ k \in N$

We will show that $(x_k)_{k \in N}$ is Cauchy sequence by estimating $d(x_k, x_{k+p})$

$d(x_k, x_{k+p}) \leq sd(x_k, x_{k+1}) + s^2 d(x_{k+1}, x_{k+2}) + s^3 d\, (x_{k+2}, x_{k+3})$

$+ \ldots + s^{p-2}\, d(x_{k+p-3}, x_{k+p-2}) + s^{p-1}\, d(x_{k+p-2}, x_{k+p-1}) + s^{p-1}\, d(x_{k+p-1}, x_{k+p})$

$d(x_k, x_{k+p}) \leq sC^k d(x_0, x_1) + s^2 C^{k+1} d(x_0, x_1) + s^3 C^{k+2} d\, (x_0, x_1)$

$+ \ldots + s^{p-2} C^{k+p-3}\, d(x_0, x_1) + s^{p-1}\, C^{k+p-2}\, d(x_0, x_1) + s^{p-1}\, C^{k+p-1} d(x_0, x_1)$

$d(x_k, x_{k+p}) \leq sC^k d(x_0, x_1)\, [I + sC + s^2 C^2 + \ldots + s^{p-2}\, C^{p-2} + s^{p-2}\, C^{p-1}]$

$d(x_k, x_{k+p}) \leq sC^k d(x_0, x_1)\, [I + sC + s^2 C^2 + \ldots + s^{p-2}\, C^{p-2} + s^{p-1}\, C^{p-1}]$

$d(x_k, x_{k+p}) \leq sC^k d(x_0, x_1)(I - sC)^{-1} \leq (sC)^k d(x_0, x_1)(I - sC)^{-1}$

I-sC is nonsingular since sC is convergent to zero,

$\Rightarrow$ The sequence $(x_k)_{k\in N}$ is d-Cauchy sequence

By (α), that $(x_k)_{k\in N}$ is a d_1- Cauchy sequence.

Since (X, d_1) is a complete generalized metric space, there exists $x^* \in X$ such that

$d_1(x_k, x^*) \to 0$ as $k \to \infty$

By (b) we have that

$d_1(Tx_{k-1}, Tx^*) \to 0$ as $k \to \infty$

But $d_1(Tx_{k-1}, Tx^*) = d_1(x_k , Tx^*)$

Hence we have

$x^* = Tx^*$.

Has x^* is a fixed point of T and

$d_1(T^k(x_0), x^*) \to 0$ as $k \to \infty$

(2) For uniqueness of fixed point

Let x^* and y be two fixed point of T, then $Tx^* = x^*$ and $Ty = y$ we claim $x^* = y$

We have

$d(x^*, y) = (Tx^*, Ty)$

$\leq \alpha\{d(x^*, Tx^*) + d(y, Ty)\} + \beta\{d(x^*, Tx^*) + d(y, Tx^*)\}$

$+ \gamma\{d(y, Ty) + d(y, Tx^*)\} + \delta\{d(x^*, Ty) + d(y, Tx^*)\}$

$\leq \alpha d(x^*, x^*) + \alpha d(y, y) + \beta d(x^*, x^*) + \beta d(y, x^*)$

$+ \gamma\, d(y, y) + \gamma d(y, x^*) + \delta d(x^*, y) + \delta d(y, x^*)$

$d(x^*, y) \leq \beta d(y, x^*) + \gamma d(y, x^*) - 2\delta d(x^*, y)$

$(I - \beta - \gamma - 2\delta)\, d(x^*, y) \leq 0$

$\Rightarrow x^* = y$

Hence x^* is the unique fixed point of T.

Since $d_1(Tx_{k-1}, Tx^*) \to 0$ as $k \to \infty$ and x^* is a unique fixed point of T,

We have

$d_1(T^k, x^*) \to 0$ as $k \to \infty$

Theorem 3.4

Let (X, d_1) be a complete generalized b-metric space and d be another vector valued b-metric on X. and for the multivalued operator $\phi : X \to pd(X)$ the following conditions are satisfied

(a) there exists

$U \in M_{n\times n}(R_+)$

such that $d_1(x, y) \leq U.\, d(x, y)$, for all $x, y \in X$;

(b) $\phi : (X, d_1) \to (p(X), 1+d_1)$ is closed;

(c) There exists matrices $\alpha, \beta, \gamma, \delta \in M_{n\times n}(R_+)$ with :

(i) $(I - \alpha - \gamma - \delta_s)$ is nonsingular and $(I - \alpha - \gamma - \delta_s)^{-1} \in M_{n\times n}(R_+)$

(ii) sC is convergent is zero, we have

$C = (I - \alpha - \gamma - \delta_s)^{-1} (\alpha + \beta + \delta_s)$;

(iii) For each $x, y \in X$ and each $u \in \phi(x)$ there exists $\upsilon \in \phi(y)$ such that

$d(u, \upsilon) \leq \alpha\{d(x, u) + d(y, u)\} + \beta\{d(x, u) + d(y, u)\}$

$+ \gamma\{d(y, \upsilon) + d(y, u)\} + \delta\{d(x, \upsilon) + d(y, u)\}$

Then ϕ has a fixed point x^* in X i.e. $x^* \in \phi(x^*)$

Proof:

Let $x_0 \in X$ and $x_1 \in \phi(x_0)$, there exists $x_2 \in \phi(x_1)$ such that

$d(x_1, x_2) \leq \alpha\{d(x_0, x_1) + d(x_1, x_2)\} + \beta\{d(x_0, x_1) + d(x_1, x_1)\}$

$+ \gamma\{d(x_1, x_2) + d(x_1, x_1)\} + \delta\{d(x_0, x_2) + d(x_1, x_1)\}$

$d(x_1, x_2) \leq \alpha d(x_0, x_1) + \alpha d(x_1, x_2) + \beta d(x_0, x_1) + \gamma d(x_1, x_2) + \delta d(x_0, x_2)$

$d(x_1, x_2) \leq \alpha d(x_0, x_1) + \alpha d(x_1, x_2) + \beta d(x_0, x_1) + \gamma d(x_1, x_2) + \delta_s d(x_0, x_1) + \delta_s d(x_1, x_2)$

$d(x_1, x_2) - \alpha d(x_1, x_2) - \gamma d(x_1, x_2) - \delta_s d(x_1,x_2) \leq \alpha d(x_0, x_1) + \beta d(x_0, x_1) + \delta_s d(x_0, x_1)$

$(I - \alpha - \gamma - \delta_s)\, d(x_1, x_2) \leq (\alpha + \beta + \delta_s)\, d(x_0, x_1)$

$d(x_1, x_2) \leq (I - \alpha - \gamma - \delta_s)^{-1} (\alpha + \beta + \delta_s)\, d(x_0, x_1)$

$d(x_1, x_2) \leq Cd(x_0, x_1)$

Now $x_2 \in \phi(x_1)$, there exists $x_3 \in \phi(x_2)$ such that

$d(x_2, x_3) \leq \alpha\{d(x_1, x_2) + d(x_2, x_3)\} + \beta\{d(x_1, x_2) + d(x_2, x_2)\}$

$+ \gamma\{d(x_2, x_3) + d(x_2, x_2)\} + \delta\{d(x_1, x_3) + d(x_2, x_2)\}$

$d(x_2, x_3) \leq \alpha d(x_1, x_2) + \alpha d(x_2, x_3) + \beta d(x_1, x_2) + \gamma d(x_2, x_3) + \delta d(x_1, x_3)$

$d(x_2, x_3) \leq \alpha d(x_1, x_2) + \alpha d(x_2, x_3) + \beta d(x_1, x_2) + \gamma d(x_2, x_3) + \delta_s d(x_1, x_2) + \delta_s d(x_2, x_3)$

$d(x_2, x_3) - \alpha d(x_2, x_3) - \gamma d(x_2, x_3) - \delta_s d(x_2, x_3) \leq \alpha d(x_1, x_2) + \beta d(x_1, x_2) + \delta_s d(x_1, x_2)$

$(I - \alpha - \gamma - \delta_s)\, d(x_2, x_3) \leq (\alpha + \beta + \delta_s)\, d(x_1, x_2)$

$d(x_2, x_3) \leq (I - \alpha - \gamma - \delta_s)^{-1} (\alpha + \beta + \delta_s)\, d(x_1, x_2)$

$d(x_2, x_3) \leq Cd(x_1, x_2)$

$d(x_2, x_3) \leq C\{Cd(x_0, x_1)\}$

$d(x_2, x_3) \leq C^2 d(x_0, x_1)$

similarly

$d(x_3, x_4) \leq C^3 d(x_0, x_1)$

By an induction method, we can construct a sequence $(x_k)_{k\in N}$ in X such that

$x_k \in \phi(x_{k-1}),\ k \in N^+$

and $d(x_k, x_{k+1}) \leq C^k d(x_0, x_1)\ \ k \in N^+$

We will prove next that $(x_k)_{k\in N}$ is Cauchy, estimating $d(x_k, x_{k+p})$.

Thus we have

$d(x_k, x_{k+p}) \leq sd(x_k, x_{k+1}) + s^2 d(x_{k+1}, x_{k+2}) + ... + s^{p-2} d\ (x_{k+p-3}, x_{k+p-2})$

$+ s^{p-1}\ d(x_{k+p-2}, x_{k+p-1}) + s^{p-1}\ d(x_{k+p-1}, x_{k+p})$

$\leq sC^k d(x_0, x_1) + s^2 C^{k+1} d(x_0, x_1) + ... + s^{p-2} C^{k+p-3} d\ (x_0, x_1)$

$+ s^{p-1} C^{k+p-2}\ d(x_0, x_1) + s^{p-1}\ C^{k+p-1} d(x_0, x_1)$

$\leq sC^k d(x_0, x_1)\ [I + sC + ... + s^{p-2} C^{p-2} + s^{p-2}\ C^{p-1}]$

$\leq sC^k d(x_0, x_1)\ [I + sC +...+ s^{p-2} C^{p-2} + s^{p-1}\ C^{p-1}]$

$\leq sC^k d(x_0, x_1)\ (I - sC)^{-1}$

$\leq (sC)^k\ d(x_0, x_1)(I - sC)^{-1}$

(I-sC) is nonsingular, since sC is convergent to zero.

$\Rightarrow$ The sequence $(x_k)_{k\in N}$ is d-Cauchy.

From (a), we get

$(x_k)_{k\in N}$ is a d_1 Cauchy sequence.

Since

(X, d_1) is a complete generalized metric space, there exists $x^* \in X$ such that

$d_1(x_k, x^*) \to 0$ as $k \to \infty$.

From (b) we get that

$x^* \in \phi(x^*)$.

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