On Roman Domination in Middle Graphs

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Abstract

Let Gbe a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. A function $f: V(G) \rightarrow \{0, 1, 2\}$ is a Roman dominating function on$G$ if for each vertex $v \in V(G)$ for which$f(v) = 0$, then there is at least one neighbouring vertex $u$such that $f(u) = 2$.The weight of a Roman domination function is the value $\sum_{v \in V(G)} |f(v)|$. The minimum weight of a Roman dominating function on $G$is called Roman domination number $\gamma_R(G)$.In this paper we initiate the study of Roman domination numbers in middle graphs. We define middle Roman dominating function and obtained some bounds related to it. In addition, we obtain middle Roman domination number for some class of graphs.

Keywords: Roman Domination number, Middle graph.

1 Introduction

Let Gbe a finite connected undirected graph with vertex set $V(G)$ and edge set $E(G)$. The cardinality of the vertex set $V(G)$ is the order and the cardinality of the edge set $E(G)$ is the size of the graph G.The open neighbourhood of $v \in V(G)$ is the set $N(v) = \{u \in V(G) | uv \in E(G)\}$ and closed neighbourhood of $v \in V(G)$ is the set $N[v] = N(v) \cup v$. The number of edges incident on the vertex $v \in V(G)$ is called degree $d(v)$ of the vertex v. The maximum and minimum degree of $G$is usually denoted by Δ and δrespectively. We write $P_n, C_n, W_n$and $K_n$for path, cycle, wheel, and complete graph, respectively. The union of two graphs $G$and $H$is written as $G \cup H$with vertex set $V(G \cup H) =$

$V(G) \cup V(H)$ and edge set $E(G \cup H) = E(G) \cup E(H)$. We refer [1] for the graph theory concepts mentioned in this paper.

T. Hamada and I. Yoshimura [2] defined middle graph of a graph. For a graph G, the middle graph of Gdenoted by $M(G)$is the graph obtained by subdividing each edge of Gexactly once and joining all the adjacent vertices of G. Precisely the vertex set, and the edge set of $M(G)$is defined as fallows, $V(M(G)) = V(G) \cup E(G)$

Two vertices $u$and $v$of $M(G)$ are adjacent if one of the following holds,

- $u, v \in E(G)$ and $u, v$are adjacent in G.
- $u \in V(G), v \in E(G)$, and u, v are incident in G.

If $G$is a graph of order $n$and size$m$then the middle graph $M(G)$ is of order $n + m$and size $2m + |E(L(G))|$.

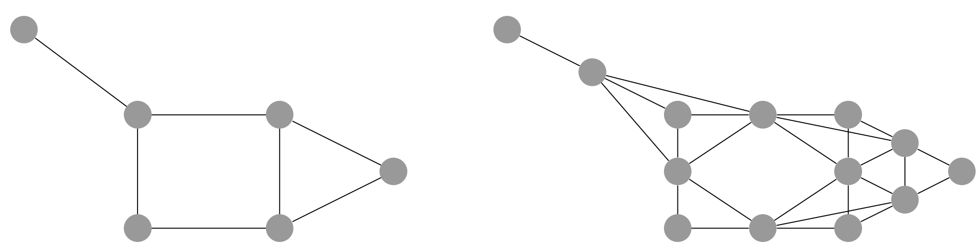

Figure 1: **Example of middle graph**

To study the Roman domination number in the class of middle graph we define the following concepts. For $v \in V(G)$, the open neighbourhood of vertex v in the middle graph $M(G)$ is defined as $\{e \in E(G) | e \text{ is incident with } u\}$ and is denoted by $N_M(v)$. For $e \in E(G)$, the open neighbourhood of an edge $e$in the middlegraph $M(G)$ is defined as$\{x \in V(G) \cup E(G) | \ x \text{ is either adjacent or incident with } e\}$ and is denoted by $N_M(e)$. The closed neighbourhood of element of Gin the middle graph of $G$is written as $N_M[x] = N_M(x) \cup \{x\}$. A middle Roman dominating function (MRDF) on a graph $G$is a function $f : V(G) \cup E(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that for every element $v \in V \cup E$for which $f(v) = 0$ is adjacent to at least one vertex $u \in V \cup E$ for which $f(u) = 2$. The weight of a middle Roman dominating function f is $w(f) = \sum_{u \in V \cup E} |f(u)|$. The minimum weight of a middle Roman dominating function of G is called middle Roman domination number $\gamma_R^*(G)$. Clearly $\gamma_R^*(G) = \gamma_R(M(G))$ for any graph G. A middle Roman dominating function can be ordered partitioned into $(V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2)$ where $V_i = \{v \in V(G) \mid f(v) = i\}$, and $E_i = \{e \in E(G) \mid f(e) = i\}$. Here weight of MRDF is $w(f) = |V_1 \cup E_1| + 2|V_2 \cup E_2|$.

Recently F. Kazemnejad et al. [6] published an article entitled “Domination Number of Middle graphs”. In this paper domination number of graphs such as star graph, double start graph, path, cycle, wheel, complete graph, complete bipartite graph, and friendship graph is computed. Also, the lower and upper bounds for the domination number of the middle graphs is obtained. Later Kijung Kim [5] introduced the middle *k*-rainbow dominating function and determined the middle 3-rainbow domination number of graphs. The bounds for the middle 3-rainbow domination number of trees are calculated in terms of the matching number.

In 2004 E. J. Cockayne et.al [3] studied the properties of Roman domination number.

Proposition 1.1. [3] For any graph $G, \gamma(G) \leq \gamma_R(G) \leq 2.\gamma(G)$.

Proposition 1.2. [3] For any graph G of order n, $\gamma(G) = \gamma_R(G)$ if and only If $G = \overline{K_n}$

Proposition 1.3. [4] If G be any graph of order n, then $\gamma_R(G) \leq n - \Delta(G) + 1$.

2 Bounds for Roman domination number of middle graphs.

In this section, we state simple propositions related to Roman domination number of middle graphs. We also derived the upper and lower bounds in terms of vertex partition of middle graph.

Proposition 2.1. For any graph $G, \gamma(M(G)) \leq \gamma_R^*(G) \leq 2.\gamma(M(G))$.

Proposition 2.2. For any graph $G$of order n, $\gamma\big(M(G)\big) = \gamma_R^*(G))$ if and only if $G = K_n$.

Proof. If $G = K_n$, obviously $\gamma\big(M(G)\big) = \gamma_R^*(G)$.

On the other hand, consider $\gamma\big(M(G)\big) = \gamma_R^*(G)$.Let us assume that Gcontains an edge $uv$and let $f$be a γ_R^*function such that $f(uv) = 2, f(u) = 0, f(v) = 0$ and $f(x) = 1$ otherwise. Obviously $\gamma\big(M(G)\big) \leq n - 1$ and$\gamma_R^*(G) \leq n$. Which is contradiction as$\gamma\big(M(G)\big) = \gamma_R^*(G))$.

Proposition 2.3. For any connected graph $G$with $p$vertices and $q$edges, $\gamma_R^*(G) \leq \frac{4(p+q)}{5}$.

Proof. Clearly the middle graph $M(G)$ is a graph with $p + q$number of vertices. Hence $\gamma_R^*(G) \leq \frac{4(p+q)}{5}$.

Proposition 2.4. Let$f = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2)$ be a γ_R^*function then,

1. $V_0 \cup E_0 \subset V_0$and $V_2 \cup E_2 \subset E_2$
2. No edge joins from $V_1 \cup E_1$ to E_2
3. The subgraph induced by $V_1 \cup E_1$ has a maximum degree 1.

Theorem 2.5. For any graph$G$, $\gamma_R^*(G) - 2 \leq 2\gamma_R^*(G+e) \leq \gamma_R^*(G) + 2$.

Proof. Let f be a γ_R^*- function of $G$and $e = uv$. Obviously, we can extend $f$to MRDF of $G + e$ by assigning $f(e) = 2$ and $f(u) = f(v) = 0$. Hence $\gamma_R^*(G + e) \leq \gamma_R^*(G) + 2$.

Let $f$be a $\gamma_R^*(G + e)$- function and $e = uv$.Now here arises two cases.

Case 1: If $f(e) = 0$, then the function $f : V(G) \cup E(G) \to \{0,1,2\}$ is a MRDF of G. This implies$\gamma_R^*(G) - 2 \leq 2\gamma_R^*(G+e) \leq \gamma_R^*(G) + 2$..

Case 2: If $f(e) \neq 0$, obviously $f(e)$ must be equal to 2. Now let us define a function $g : V(G) \cup E(G) \to \{0,1,2\}$ such that $g(u) = g(v) = 0$ and $g(x) = f(x) \forall x \in V(G) \cup E(G)$. Now $g$is a MRDF with weight$\gamma_R^*(G+e) + f(e)$. Thus$\gamma_R^*(G) - 2 \leq 2\gamma_R^*(G+e) \leq \gamma_R^*(G) + 2$.

Theorem 2.6. For any graph $G$with order $n \geq 2$ and $v \in V(G))$,

$$\gamma_R^*(G\backslash v) \leq \gamma_R^*(G) + 2.$$

Proof. Let f be a γ_R^*-function of $G\backslash v$. Clearly, we can extend f to MRDF of G by assigning $f(v) = 2$ and $f(e_i) = 0$ each e_i is incident with vertex v.

Hence

$$\gamma_R^*(G\backslash v) \leq \gamma_R^*(G) + 2.$$

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Theorem 2.7. For any graph G with maximum degree $\Delta \geq 2$,

$$\gamma_R^*(G) - \Delta(G) + 1 \leq \gamma_R^*(G \backslash v) \leq \gamma_R^*(G).$$

Proof. Let f be a γ_R^*- function of $G\backslash v$. We define a function $g : V(G) \cup E(G) \rightarrow \{0,1,2\}$ such that $g(v) = 2$, $g(e_i) = 0$ for all $e_i \in N_M(v)$ and $g(u) = f(u)$ otherwise. Clearly function g is a MRDF of G with weight at most $\gamma_R^* - \Delta(G) + 1$.Hence $\gamma_R^*(G) - \Delta(G) + 1 \leq \gamma_R^*(G\backslash v)$.

Now we need to prove that $\gamma_R^*(G\backslash v) \leq \gamma_R^*(G)$. Let f be a γ_R^*- function of G. Let us define a function $h : V(G \backslash v) \cup E(G\ v) \rightarrow \{0,1,2\}$ by $h(u) = 1$ if $u \in N_M(v)$ and $h(w) = f(w)$ otherwise. Clearly h is MRDG of G with a weight greater that weight of f.

Theorem 2.8. For any connected graph G on n vertices and m edges, $\gamma_R^*(G) \leq (m + n) - \Delta(G) + 1$.

Theorem 2.9. For any graph G of order n and size m,

$$5 \leq \gamma_R^*(G) + \gamma_R^*(\bar{G}) \leq \frac{n(n+1)}{2} + 3.$$

Proof. If G is a graph having at least two vertices, then $\gamma_R^*(G) \geq 2$. The equality holds, only when $M(G)$ has a dominating vertex. Since, the graph G and its complement G does not contain domination vertices, we have $\gamma_R^*(G) + \gamma_R^*(\bar{G}) \geq 5$.Equality holds if and only if G or $\bar{G}$ has an edge $e = uv$ with $d(u) + d(v) = n - 1$ and its complement has an edge $e' = xy$ with $d(x) + d(y) = n - 2$.

Now we claim that $\gamma_R^*(G) + \gamma_R^*(\bar{G}) \leq m + n + 3$. From the above theorem 2.8,

$$\gamma_R^*(G) + \gamma_R^*(\bar{G}) \leq (n + m - \Delta(G) + 1) + \left(\frac{n(n-1)}{2} - m - \Delta(\bar{G}) + 1\right)$$

$$= n + \frac{n(n-1)}{2} - \Delta(G) + \delta(G) + 3$$

$$\leq \frac{n(n+1)}{2} + 3.$$

3 Middle Roman domination number of some classes of graphs

In this section we calculated middle Roman domination number for star graph, path, cycle, complete bipartite graph, and friendship graph.

Proposition 3.1. For any star graph $K_{1,n}$ on $n + 1 \geq 2$ vertices,

$$\gamma_R^*(K_{1,n}) = n + 1$$

Proof. Let us consider the vertex set $V(K_{1,n}) = \{v_0, v_1, \ldots, v_n\}$ and edge set $E(K_{1,n}) = \{v_0v_1, v_0v_2, \ldots, v_0v_n\}$. Then $V(M(K_{1,n}1)) = V(K_{1,n}1) \cup A$ where $A = \{a_i, 1 \leq i \leq n\}$ and $E(M(K_{1,n})) = \{a_1v_1, a_2v_2, \ldots, a_nv_n\} \cup X \cup Y$ where $X = \{v_0a_i, 1 \leq i \leq n\}, Y = \{a_ia_j, 1 \leq i \leq n, 1 \leq j \leq n\}$.
Clearly the vertices $\{v_0, a_1, a_2, \ldots a_n\}$ forms a complete graph. Let us define a γ_R^* - function f such that $V_0 \cup E_0 = \{v_0, v_1\} \cup \{e_i, 2 \leq i \leq n\}, V_1 \cup E_1 = \{v_i, 2 \leq i \leq n\}$ and $V_2 \cup E_2 = \{e_1\}$. Hence

$$\gamma_R^*(K_{1,n}) = |V_1 \cup E_1| + 2|V_2 \cup E_2| = n - 1 + 2 = n + 1.$$

Definition 3.1. A double star graph $S_{1,n,n}$ is obtained from the star graph $K_{1,n}$ by replacing every edge with a path of length 2.

Proposition 3.2. For any double star graph $S_{1,n,n}$ on $2n + 1$ vertices with $n \geq 2$,

$$\gamma_R^*(S_{1,n,n}) = 2n + 1.$$

Proposition 3.3. For a path of length $n \geq 2$,

$$\gamma_R^*(P_n) = n + 1.$$

Proof. Let us consider a path of length n i.e., $P_n = v_1v_2v_3 \ldots v_n$. Divide each edge $v_1v_2, v_2v_3, v_3v_4, \ldots, v_{n-1}v_n$ by introducing new vertices $x_1, x_2, x_3, \ldots, x_{n-1}$ respectively. Now we obtain a path $P_{2n-1} = v_1x_1v_2x_2v_3x_3 \ldots x_{n-1}v_n$ of length $2n - 1$. Clearly the middle graph $M(P_n) = P_{2n-1} + Q_{n-1}$ where $Q_{n-1} = x_1x_2x_3 \ldots x_{n-1}$. We prove the result by induction on n. One can easily obtain $\gamma_R^*(P_3) = 4, \gamma_R^*(P_4) = 5, \gamma_R^*(P_5) = 6$.
Let us assume the result holds for $n = m$. We claim that $\gamma_R^*(P_{m+1}) = m + 2$.

Consider a Roman dominating function $f : V(P_{m+1}) \cup E(P_{m+1}) \rightarrow \{0,1,2\}$ with a vertex partition $V_0 \cup E_0 = x_1, v_2, v_3, x_3, v_4, x_4, \ldots\}, V_1 \cup E_1 = \{v_1, v_4, v_8, \ldots\}$ and $V_2 \cup E_2 = \{x_2, x_5, x_8, \ldots\}$. Then, the Roman domination number is $\gamma_R^*(P_{m+1}) = \gamma_R^*(P_m) + f(v_{m+1}) + f(x_m)$. Suppose $x_{m-1} \in V_2 \cup E_2$, then clearly $f(x_m) = 0$ and $f(v_{m+1}) = 1$. On the other hand, if $x_{m-1} \in V_0 \cup E_0$ then obviously $f(v_m) = 1$. Now assign $f(x_m) = 2\, f(v_m) = 0$ and $f(v_{m+1}) = 0$. Hence the proof.

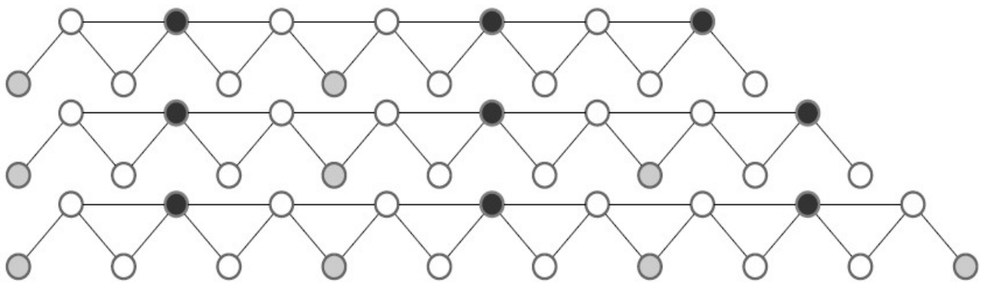

Figure 2: γ_R^***function for path** P_7, P_8**and** P_9

In figure 2, the vertices assigned with 0, 1 and 2 are shaded by white, grey, and black colours respectively.

Proposition 3.4. For a cycle of length$n \geq 3, \gamma_R^*(C_n) = n$.

Proof. Let $C_n = v_1 v_2 \ldots v_n v_1$be a cycle of length n. Let $f$be a γ_R^*function of a path P_n with minimum weight. Clearly the $E(M(C_n)) = E(M(P_n)) \cup \{v_1 x_n, x_n v_n\}$. Here, we get two cycles $C_{2n} = v_1 x_1 v_2 x_2 \ldots v_n x_n$ and $C_n = x_1 x_2 \ldots x_n$. Hence the middle graph $M(C_n) = C_{2n} \cup C_n$. Now let f be a γ_R^*function for the path $P_n$and we can extend this function for the cycle $C_n$and $C_n$is a closed path of length n. Hence $\gamma_R^*(C_n) = n$.

Corollary 3.5. For any wheel graph $W_n$where$n \geq 3$, $\gamma_R^*(W_n) = n - 1$.

Proposition 3.6. For any complete bipartite graph $K_{m,n}$with $1 \leq m \leq n$,

$$\gamma_R^*(K_{m,n}) = m + n.$$

Proof. Let us assume that,$V(K_{m,n}) = \{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_n\}$ and $E(K_{m,n}) = \{v_i u_j = x_{ij} \,|\, 1 \leq i \leq m, 1 \leq j \leq n\}$. Clearly,

$V\left(M(K_{m,n})\right) = V(K_{m,n}) \cup E(K_{m,n})$ and $E\left(M(K_{m,n})\right) = E(K_m) \cup \{v_i x_{ij} | 1 \le i \le m \text{ and } 1 \le j \le n\}\{u_j x_{ij} | 1 \le i \le m \text{ and } 1 \le j \le n\}$. Let us define $f : V(K_{m,n}) \cup E(K_{m,n}) \to \{0,1,2\}$ with minimum weight by $f(x_{ij}) = 2 \text{ for } i = j$, $f\left(N_M(x_{ij})\right) = 0$ and $f(u_j) = 1$ for $n - m \le j \le n$. Hence$\gamma_R^*(K_{m,n}) = 2m + (n - m) = m + n$.

Definition 3.2. The friendship graph $F_n$of order $2n + 1$ is obtained by joining $n$copies of the cycle $C_3$with a common vertex.

Proposition 3.7. Let $F_n$be a friendship graph with $n \ge 2$ then,

$$\gamma_R^*(F_n) = 2n + 1.$$

Proof. Let vertex set $V(F_n) = \{v_0, v_1, \ldots, v_{2n}\}$ and edge set $E(F_n) = \{e_i = v_0 v_i | 1 \le i \le 2n\} \cup \{v_1 v_2, v_3 v_4, \ldots, v_{2n-1} v_{2n}\}$. Clearly the middle graph of $F_n$form a complete graph $K_{2n+1}$with the vertex set $v_0 \cup \{e_i = v_0 v_i | 1 \le i \le 2n\}$.

Hence,

$$\gamma_R^*(F_n) = \gamma_R(K_{2n+1}) + (n-1)\gamma_R(C_3) + 1$$

$$= 2n + 1$$

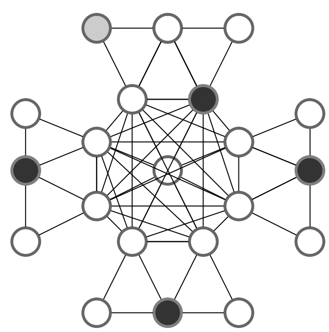

Figure 3:γ_R^***function for the friendship graph F_4**.

Definition 3.3. [6] The corona graph $G \circ K_1$also denoted by $cor(G)$, of a graph $G$is the graph of order $2 | V(G) |$ obtained by adding a pendent edge to each vertex of G. The 2-corona $G \circ P_2$of $G$is the graph of order $3 | V(G) |$ obtained by adding a path of length 2 to each vertex of G.

Theorem 3.8. For any connected graph G of order $n \geq 2$,

$$\gamma_R^*(G \circ K_1) \leq 2n.$$

Proof. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Clearly $V(G \circ K_1) = \{v_1, v_2, \ldots, v_{2n}\}$ and $E(G \circ K1) = \{v_1v_{n+1}, \ldots, v_nv_{2n}\} \cup E(G)$. Then $V(M(G \circ K_1)) = V(G \circ K_1) \cup \mathcal{M} \cup \mathcal{A}$, Where $\mathcal{M} = \{m_{i(n+i)}) \mid 1 \leq i \leq n\}$ and $\mathcal{A} = \cup\{a_{ij} \mid v_iv_j \in E(G)\}$.

The MRDF $(\mathcal{A}, \emptyset, \mathcal{M})$ has weight $2|\mathcal{M}| = 2n$, and hence

$$\gamma_R^*(G \circ K_1) \leq 2n.$$

4 Conclusion

We conclude this paper, by posing the following open problems.

Problem 1. For any $\gamma_R^*(G)$- function $f = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2)$ of a connected graph G of order $n \geq 3$ and size m, obtain upper and lower bounds for $|V_0 \cup E_0|, |V_1 \cup E_1|, |V_2 \cup E_2|$.

Problem 2. Find Middle Roman domination number for other class of graphs like n-partite graph, generalized Peterson graph $P(n,2)\, P(n,3)$, circulant graphs.

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