Fixed Point Theorem for Multi-valued $k-$Strictly Pseudo-contractive Mapping on Hilbert Space

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Abstract

In this paper, we introduced new hybrid iterative scheme which known as Picard-Mann iterative scheme. Moreover, the strong convergence results of xed point for multivalued $k-$strictly pseudo-contractive mapping in Hilbert space using Picard-Mann iteration scheme.

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Key Words: Hilbert space, $k-$strictly pseudo-contractive mappings, xed point theorem, Picard-Mann iterative scheme.

1 Introduction

Let N be a normed linear space and let K be a non-empty subset of N. Then K is said to be proximinal if (see, e.g.[11], [14], [16], [17]) if for each $u \in N$, there exists $x \in K$ such that

$$\|u - x\| = d(u, K) = inf\{\|u - v\| : v \in K\} \tag{1}$$

It is known that every non-empty closed, convex subset of a real Hilbert space is proximinal. Let $P(K)$ is the family of non-empty proximinal bounded

subsets of K and $CB(K)$ is the family of non-empty closed bounded subsets of K. The Hausdor metric on $CB(K)$ is de ned by

$$H(A,B) = max \left\{ \sup_{u \in A} d(u,B), \sup_{v \in B} d(A,v) \right\}, \quad \forall A, B \in CB(K) \tag{2}$$

De nition 1.1. Let K be a nonempty closed, convex subset of a real Hilbert space H and let $T : K \to CB(K)$ be a multi-valued mapping on H. Then T is said to be non-expansive mapping on H if

$$(H(Tu, Tv))^2 \leq \|u - v\|^2, \quad \forall u, v \in K \tag{3}$$

De nition 1.2. Let K be a nonempty closed, convex subset of a real Hilbert space H and let $T : K \to CB(K)$ be a multi-valued mapping on H. Then T is said to be quasi-nonexpansive mapping on H if

$$(H(Tu, q))^2 \leq \|u - q\|^2, \quad \forall u \in K \tag{4}$$

where, q is the xed point of T.

Recall that every nonexpansive mapping with non-empty xed point set is quasi-nonexpansive mapping.

De nition 1.3. Let K be a nonempty closed, convex subset of a real Hilbert space H and let $T : K \to CB(K)$ be a multi-valued mapping on H. Then T is said to be $k-$strictly pseudo-contractive mapping on H if there exists $k \in (0, 1)$ such that

$$(H(Tu, Tv))^2 \leq \|u - v\|^2 + k\|(I - T)u - (I - T)v\|^2, \quad \forall u, v \in K \tag{5}$$

where I is identity mapping.

If $k = 1$ in (5), then T is pseudo-contractive Mapping. The class of

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nonexpansive maps is proper sub-class of the class of $k-$strictly pseudo-contractive mappings.

By de nitions, we observe that every multi-valued nonexpansive is $k-$strictly pseudo-contractive mapping but converse may not true, as can be seen from the following example.

Example 1. Let $T : [0, 1] \rightarrow CB(R)$ is de ned by $Tu = \{0, 5 - \frac{5}{4}u\}$. Then, we have

$$H(Tu, Tv) = max \left\{ \sup_{x \in Tu} d(x, Tv), \sup_{y \in Tv} d(y, Tv) \right\}$$

$$= max\{min(|5 - \frac{5}{4}u|, \frac{5}{4}|u - v|), min(|5 - \frac{5}{4}v|, \frac{5}{4}|u = v|)\}$$

$$= \frac{5}{4}|u - v|$$

Hence,
$$(H(Tu, Tv))^2 = |u - v|^2 + \frac{9}{16}|u - v|^2$$

Obiously, T is not expansive-type. we have to show that it is $k-$strictly pseudocontractive mapping, with out lose of generality assume that $u < v$. we will take four cases.

Case-1: Let $x = 0$ and $y = 0$. Then $|u - v - (x - y)| = |u - v|$ and hence

$$(H(Tu, Tv))^2 \leq |u - v|^2 + \frac{9}{16}|u - v - (x - y)|^2$$

Case-2: Let $x = 5 - \frac{5}{4}u$ and $y = 0$. Then $u - v - (5 - \frac{5}{4}u) \leq u - v \leq 0$.
Thus $|u - v - (5 - \frac{5}{4}u - 0)|^2 = |u - v - (5 - \frac{5}{4}u)|^2 \geq |u - v|^2$

Hence

$$(H(Tu, Tv))^2 \leq |u - v|^2 + \frac{9}{16}|u - v - (x - y)|^2$$

Case-3: Let $y = 5 - \frac{5}{4}v$ and $x = 0$. Then $5 - \frac{5}{4}v \geq 2(v - u)$ and

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$u - v - (5 - \frac{5}{4}v) = u - v + 2(v - u) \geq v - u \geq 0.$
Thus $|u - v - (0 - (5 - \frac{5}{4}v))|^2 = |u - v - (5 - \frac{5}{4}v)|^2 \geq |u - v|^2$

Hence

$$(H(Tu, Tv))^2 \leq |u - v|^2 + \frac{9}{16}|u - v - (x - y)|^2$$

Case-4: Let $x = 5 - \frac{5}{4}u$ and $y = 5 - \frac{5}{4}v$.
Then $|(u - v) - (5 - \frac{5}{4}u - (5 - \frac{5}{4}v))|^2 = |(u - v) + \frac{5}{4}(u - v|^2 \geq |u - v|^2$

Hence

$$(H(Tu, Tv))^2 \leq |u - v|^2 + \frac{9}{16}|u - v - (x - y)|^2$$

Therefore, T is k-strictly pseudocontractive mapping.

Let K be a closed convex subset of a Hilbert space H and let $T : K \to CB(K)$ be a multivalued mapping. Suppose that $F_T = \{q \in K : Tq = q\}$ is the set of xed points of T.

In Picard iterative scheme $u_1 \in K$ and $\{u_n\}_{n=1}^{\infty}$ de ned by

$$u_{n+1} = Tu_n, \tag{6}$$

In 1953, W. R. Mann [M] introduced the following iterative scheme, for $u_1 \in K$ and $\{u_n\}_{n=1}^{\infty}$ de ned by

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n Tu_n \tag{7}$$

where $\{\alpha_n\}$ is sequence of positive numbers in $(0, 1)$ and it is called Mann iterative scheme.

Now we introduced Picard-Mann hybrid iteration scheme, for $u_1 \in K$ and

$\{u_n\}_{n=1}^{\infty}$ de ned by

$$u_{n+1} = y_n,\ y_n \in Tv_n$$

$$v_n = (1 - \alpha_n)u_n + \alpha_n x_n, \quad x_n \in Tu_n \tag{8}$$

where α_n, β_n are sequences of positive numbers satisfying

(i) $0 \le \alpha_n \le 1,\ n \ge 1$; (ii) $\lim_{n\to\infty} \beta_n = 0$; (iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

The study of xed point theorem for multi-valued nonexpansive mappings using the Hausdor metric by several authors (see, e.g., Markin [9], Nadler Jr.[10] and Lim [7]), the theory of developed greatly with applications in control theory, di erential inclusion, convex optimization and economics. In 2005, Sastry and Babu [14] introdeced Mann and Ishikawa iteration schemes for multi-valued nonexpansive mappings. Panyanak [11], Song and Wang [16], [17] extended the result of Sastry and Babu [14]. Shahzad and Zegeye [15] has exploting the results of Panyanak [11], Song and Wang [16], and Sastry and Babu [14] to multivalued quasi-nonexpansive maps. Also, the restriction $Tq = \{q\},\ \forall q \in F(T)$ remove in Theorem Song and Wang, they introduced a new iteration scheme as follows

Let K be a non-empty closed-convex subset of a real Banach space and let $T : K \to P(K)$ be a multivalued map, and $P_T u = \{v \in Tu : \|u - v\| = d(u, Tu)\}$. Choose $u_1 \in K$ and $\{u_n\}_{n=1}^{\infty}$ de ned by

$$u_{n+1} = (1 - \alpha_n)\, u_n + \alpha_n x_n$$

where $x_n \in P_T u_n$ and $\{\alpha_n\}$ is sequence of positive numbers in $[0, 1]$.

In 2013, Chidume et al. [2], obtained convergence theorems for xed points of multivalued strictly pseudo-contractive mappings in Hilbert spaces as follows:

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Theorem 1.1. [2] Let K be a non-empty closed, convex subset of a real Hilbert space H and let $T : K \to CB(K)$ be a multivalued $k-$strictly pseudo-contractive mapping such that $F(T) \neq \emptyset$. Assume that $Tq = q$, $\forall q \in F(T)$. Let $\{u_n\}$ be a sequence defined by $u_1 \in K$,

$$u_{n+1} = (1-\lambda)u_n + \lambda x_n \tag{9}$$

where $x_n \in Tu_n$ and $\lambda \in (0, 1-k)$. Then $\lim_{n\to\infty} d(u_n, Tu_n) = 0$.

Theorem 1.2. [2] Let K be a non-empty closed, convex subset of a real Hilbert space H and let $T : K \to CB(K)$ be a multivalued $k-$strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ such that $Tq = q$, $\forall q \in F(T)$. Suppose that T is continuous. Let $\{u_n\}$ be a sequence defined by $u_1 \in K$,

$$u_{n+1} = (1-\lambda)u_n + \lambda x_n \tag{10}$$

where $x_n \in Tu_n$ and $\lambda \in (0, 1-k)$. Then the sequence converges strongly to a fixed point q of T.

Motivated by the work of Chidume et al. [2], Shahzad and Zegeye [15], Panyanak [11], Song and Wang [16], [17], and Sastry and Babu [14]. In this paper, we have studied the results of convergence by Picard-Mann iterative scheme for multi-valued strictly pseudo-contractive Mapping in Hilbert spaces.

2 Preliminaries

Lemma 2.1. [12] Let H be a real Hilbert space. Then $\forall x, y \in H$, $\lambda \in [0, 1]$

$$\|\lambda x - (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{11}$$

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Lemma 2.2. [3] Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers satisfying the following condition:

$$a_{n+1} \leq a_n + b_n \tag{12}$$

where, $\forall\, n \geq n_0$ and n_0 is natural number. If $\sum b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Definition 2.1. [13] Let $F(T)$ is the non-empty set of fixed points of T. Then T is said to satisfy Tricomi condition if for each $x \in K$ and every $q \in F(T)$ such that

$$H(Tu, q) \leq \|u - q\| \tag{13}$$

where, q is the fixed point of T.

3 Our Results

Theorem 3.1. Let K be a non-empty closed, convex subset of a real Hilbert space H and let $T : K \to CB(K)$ be a $k-$strictly pseudo-contractive mapping satisfying the Tricomi condition (13) and $F(T) \neq \emptyset$ and $Tq = q$, $\forall\, q \in F(T)$. Suppose for any $u_1 \in K$, let $\{u_n\}_{n=1}^{\infty}$ be a sequence defined by (8). Then

$$\lim_{n\to\infty} \inf d(u_n, Tu_n) = 0.$$

Proof. Since T is a strictly pseudo-contractive mapping, for any $u, v \in K$ we have

$$(H(Tu, Tv))^2 \leq \|u - v\|^2 + k\|(I - T)u - (I - T)v\|^2 \tag{14}$$

where, I is a identity mapping.

Let $F(T) \neq \emptyset$ with $Tq = q$, $\forall\, q \in F(T)$. Recall Lemma 2.1 for any $u, v, , w$

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in a Hilbert space H and a real number λ, we have

$$\|\lambda u + (1 - \lambda)v - w\| = \lambda\|u - w\| + (1 - \lambda)\,\|v - w\| - \lambda(1 - \lambda)\,\|u - v\| \quad (15)$$

Using (15) we obtain the following equalities

$$\begin{aligned}\|u_{n+1} - q\|^2 &= \|y_n - q\|^2 \\ &\le \{H(Tv_n, q)\}^2 \\ &\le \|v_n - q\|^2 \end{aligned} \quad (16)$$

Now (8) and (15), we have

$$\begin{aligned}\|v_n - q\|^2 &= \|(1 - \alpha_n)u_n + \alpha_n x_n - q\|^2 \\ &= (1 - \alpha_n)\|u_n - q\|^2 + \alpha_n\|x_n - q\|^2 - \alpha_n(1 - \alpha_n)\|u_n - x_n\|^2 \\ &\le (1 - \alpha_n)\|u_n - q\|^2 + \alpha_n\{H(Tu_n, q)\}^2 - \alpha_n(1 - \alpha_n)\|u_n - x_n\|^2 \\ &\le (1 - \alpha_n)\|u_n - q\|^2 + \alpha_n(\|u_n - q\|^2 \\ &\quad + k\|(I - T)u_n - (I - T)q\|^2) - \alpha_n(1 - \alpha_n)\|u_n - x_n\|^2 \\ &= \|u_n - q\|^2 - \alpha_n(1 - \alpha_n - k)\|u_n - x_n\|^2 \end{aligned} \quad (17)$$

Putting equation (17) in (16), we get

$$\|u_{n+1} - q\|^2 \le \|u_n - q\|^2 - \alpha_n(1 - \alpha_n - k)\|u_n - x_n\|^2 \quad (18)$$

By summing (20) for $n \in \{m, m + 1, m + 2, \dots, n\}$, we obtain

$$\|u_{n+1} - q\|^2 \le \|u_m - q\|^2 - \sum_{j=m}^{n} \alpha_j(1 - \alpha_j - k)\|u_j - x_j\|^2$$

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which can be written as

$$\sum_{j=m}^{n} \alpha_j(1 - \alpha_j - k)\|u_j - x_j\|^2 \le \|u_m - q\|^2 - \|u_{n+1} - q\|^2$$

Since, $0 \le k < 1$ i.e., $1 - k > 0$. Therefore, by expoliting the assumption

(ii) , we deduce that there exists a positive integer N such that

$$\alpha_j \le \frac{1-k}{2}, \text{ for all integers } k \ge N$$

Then, for $m > N$, we obtain

$$\left(\frac{1-k}{2}\right)\sum_{j=m}^{n} \alpha_j \|u_j - x_j\|^2 \le \|u_m - q\|^2 - \|u_{n+1} - q\|^2 \quad (19)$$

Since K is bounded, the right-hand side quantity in (19) is bounded. This means that the series in the left-hand side is convergent and therefore, by assumption (iii), it results that $\lim_{n\to\infty} \inf \|u_n - x_n\| = 0$.

Since $x_n \in Tu_n$, therefore $d(u_n, Tu_n) \le \|u_n - x_n\|$

Hence, $\lim_{n\to\infty} \inf d(u_n, Tu_n) = 0$. □

Theorem 3.2. Let K be a non-empty closed, convex subset of a real Hilbert space H and let $T : K \to CB(K)$ be a k-strictly pseudo-contractive mapping satisfying the Tricomi condition (13) and $F(T) \ne \emptyset$ and $Tq = q$, $\forall q \in F(T)$. Suppose for any $u_1 \in K$, let $\{u_n\}_{n=1}^{\infty}$ be a sequence defined by (8). Then the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to fixed point of T

Proof. From Theorem 3.1, we have that $\lim_{n\to\infty} \inf d(u_n, Tu_n) = 0$. So there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $\|u_{n_j} - q_j\| \le \frac{1}{2^k}$, for some $\{q_j\} \subset F(T)$.

From equation (8), we have

$$u_{n_{j+1}} - q_j \leq u_{n_j} - q_j \tag{20}$$

Now we have to show that $\{q_j\}$ is a cauchy sequence in $F(T)$. We have

$$\|q_{j+1} - q_j\| \leq q_{j+1} - u_{n_{j+1}} + u_{n_{j+1}} - q_j$$
$$\leq \frac{1}{2^{k+1}} + \frac{1}{2^k}$$
$$= \frac{1}{2^{k-1}}$$

Therefore, $\{q_j\}$ is a cauchy sequence in $F(T)$ and converges to some $q \in K$, because K is closed. So,

$$u_{n_{j+1}} - q \leq u_{n_{j+1}} - q_j + \|q_j - q\|$$

Hence $u_{n_j} \longrightarrow q$ as $j \longrightarrow \infty$. We have Since $u_{n_j} \longrightarrow q$ and T being continous

$Tu_{n_j} \longrightarrow Tq$. Therefore, for given $s > 0$, there exists we choose n_j such that

$$d(u_{n_j}, q) < \frac{s}{3}, \quad d(Tu_{n_j}, Tq) < \frac{s}{3} \text{ and } d(u_{n_j}, Tu_{n_j}) < \frac{s}{3}$$

Now,

$$d(Tq, q) \leq d(Tq, Tu_{n_j}) + d(Tu_{n_j}, u_{n_j}) + d(u_{n_j}, q)$$
$$< \frac{s}{3} + \frac{s}{3} + \frac{s}{3} = s$$

Since q is a xed point of T, from (20), we obtain for $n \geq N$

$$\|u_{n+1} - q\|^2 \leq \|u_n - q\|^2$$

that is, the sequence $\{\|u_n - q\|\}$ is non-increasing. Having in view that there is a subsequence $\{ u_{n_j} - q \}$ converging to zero. It fanally results that Then the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to xed point q of T □

Corollary 3.1. Let K be a non-empty convex, compact subset of a real Hilbert space H and let $T : K \to CB(K)$ be a k-strictly pseudo-contractive mapping satis ng the Tricomi condition (13) and $F(T) \neq \emptyset$ and $Tq = q, \ \forall q \in F(T)$. Suppose for any $u_1 \in K$, let $\{u_n\}_{n=1}^{\infty}$ be a sequence de ned by (8). Then the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to xed point q of T

Proof. From Theorem 3.1, we have that $\lim_{n\to\infty} \inf d(u_n, Tu_n) = 0$. So there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $\lim_{n\to\infty} \inf d(u_{n_j}, Tu_{n_j}) = 0$. Since, K is compact. Therefore, there is the subsequence $\{u_{n_j}\}$ that converges to a certain point q of $F(T)$. We have to show that q is a xed point of T. Since $u_{n_j} \longrightarrow q$ and T being continous $T_{u_{n_j}} \longrightarrow Tq$. Therefore, for given $s > 0$, there exists we choose n_j such that

$$d(u_{n_j}, q) < \frac{s}{3}, \ d(Tu_{n_j}, Tq) < \frac{s}{3} \text{ and } d(u_{n_j}, Tu_{n_j}) < \frac{s}{3}$$

Now,

$$\begin{aligned} d(Tq, q) &\le d(Tq, Tu_{n_j}) + d(Tu_{n_j}, u_{n_j}) + d(u_{n_j}, q) \\ &< \frac{s}{3} + \frac{s}{3} + \frac{s}{3} = s \end{aligned}$$

Since q is a xed point of T, from (20), we obtain for $n \ge N$

$$\|u_{n+1} - q\|^2 \le \|u_n - q\|^2$$

that is, the sequence $\{\|u_n - q\|\}$ is non-increasing. Having in view that there is a subsequence $\{ u_{n_j} - q \}$ converging to zero. It fanally results that Then the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to xed point q of T □

Competing interests

The authors declare that they have no competing interests.

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