A study on multivariable I-Function and its summation formula

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Abstract
In this paper we establish a summation formula for the I- function of 'r' variables
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1. Introduction

: The I – function of several complex variable is defined as follows

$$\overline{I}\begin{bmatrix} z_1 \\ z_r \end{bmatrix} = I^{0,n:m_1,n_1;m_r,n_r}_{p,q:p_1,q_1;p_r,q_r}\left[\begin{matrix} z_1 \\ z_r \end{matrix}\middle| \begin{matrix} \left(a_j;\alpha_j^{(1)},\ldots\ldots,\alpha_j^{(1)};A_j\right)_{1,p};\left(c_j^{(1)},\ldots,\gamma_j^{(1)};C_j\right)_{1,p_1};\ldots;\left(c_j^{(r)},\ldots,\gamma_j^{(r)};C^{(r)}{}_j\right)_{1,p_1} \\ \left(b_j;\beta_j^{(1)},\ldots\ldots,\beta_j^{(1)};B_j\right)_{1,q};\left(d_j^{(1)},\ldots,\delta_j^{(1')};D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\ldots,\delta_j^{(r')};D_j^{(r)}\right)_{1,q_r} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\ldots\ldots\int_{L_r}\phi(s_1,\ldots,s_r)\ \overline{\theta_1}(s_1)\ldots\overline{\theta_r}(s_r)\ z_1^{s_1}\ldots z_r^{s_r}\quad ds_1\ldots\ldots ds_r, \tag{1.1}$$

Where

$$\overline{\theta_1}(s_1) = \prod_{j=1}^{n_j}\Gamma^{C_j^{(i)}}\left(1-c_j^{(1)}+\gamma_j^{(1)}s_i\right)\prod_{j=1}^{m_j}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}s_i\right)$$

$$\left(\prod_{j=n_i+1}^{n_j}\Gamma^{C_j^{(i)}}\left(1-c_j^{(1)}+\gamma_j^{(1)}s_i\right)\prod_{j=1}^{m_j}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}-\delta_j^{(i)}s_i\right)\right)^{-1}$$

where i=1,...,r.

$$\phi(s_1,\ldots,s_r) = \frac{1}{\prod_{j=1}^{p}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^i s_i\right)\prod_{j=1}^{q}\Gamma^{B_j}\left(1-b_j+\sum_{i=1}^{r}\beta_j^i s_i\right)} \tag{1.2}$$

$$\overline{\theta_1}(s_1) = \frac{\prod_{j=1}^{n_j}\Gamma^{C_j^{(i)}}\left(1-c_j^{(1)}+\gamma_j^{(1)}s_i\right)\prod_{j=1}^{m_j}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}s_i\right)}{\prod_{j=n_1+1}^{p_j}\Gamma^{C_j^{(i)}}\left(c_j^{(1)}-\gamma_j^{(1)}s_i\right)\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}-\delta_j^{(i)}s_i\right)}$$

$$\forall i = 1,\ldots.r. \tag{1.3}$$

2. Results Used :

To facilitate the derivation of our main result, we shall require the following results.

(a)Richard Askey[7]:

$$(sin\theta)^{1-2u}\, P_n^{1-u}(cos\theta) = \sum_{k=0}^{\infty} \frac{2^{2u}\ (n+k)!\,\Gamma(n+2-2u)\,\Gamma(k+u)\ sin(n+2k+1)\theta}{\Gamma(1-u)\,\Gamma(u)\ k!\ n!\,\Gamma(n+k+2-u)}$$

(2.1)

provided $u < 1$, $0 \le \theta \le \pi$ and 'n' is a non negative integer.

(b)Rainville [6] :

$$(i) \qquad \sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right)$$

(2.2)

$$(ii) \qquad P_n^{\mu}(z) = \sum_{m=0}^{n} \frac{(2\mu)_{n+m}\,(z-1)^m}{2^m\ m!\ (n-m)!\ (\mu+\frac{1}{2})_m}$$

(2.3)

where $(a)_n$ is the pochhammer symbol defined by,

$(a)_n = a(a+1)(a+2)....(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$ and $(a)_0 = 1$ **(2.4)**

(c) Srivastava , Gupta and Goyal [10]:

$$(i) \qquad P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\ {}_2F_1\left[-n, 1+\alpha+\beta+n\,; 1+\alpha\,; \frac{1-x}{2}\right]$$

(2.5)

$$(ii) \qquad U_n(x) = \frac{sin[(n+1)cos^{-1}x]}{sin(cos^{-1}x)}\ ;\ n \geq 0$$

(2.6)

$$sinnz = n\ sinz\ {}_2F_1\left[\frac{1-n}{2}, \frac{1+n}{2}; \frac{3}{2}; sin^2 z\right]\ ; |sinz| < 1$$

$$(iii) \qquad U_n(x) = \frac{(n+1)!}{\left(\frac{3}{2}\right)_n} P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(x)$$

(2.7)

(d) Erdelyi [1] :

$$sinnz = n\ sinz\ {}_2F_1\left[\frac{1-n}{2}, \frac{1+n}{2}; \frac{3}{2}; sin^2 z\right]\ ; |sinz| < 1$$

(2**.8)**

3. Summation Formula:

$$\frac{\sqrt{\pi}}{2}(sin\theta)^{2\nu-1}\sum_{m=0}^{n}\left[\frac{(cos\theta-1)^m}{2^m\ m!\ (n-m)!}\right]$$

$$I^{0,n:m_1,n_1;m_r,n_r}_{p,q:p_1,q_1;p_r,q_r}\left[\begin{array}{c} z_1 \\ z_r \end{array}\middle|\begin{array}{l}(a_j;\alpha_j^{(1)},\ldots\ldots,\alpha_j^{(1)};A_j)_{1,p};(\frac{1}{2}+\nu+m,\ldots,\lambda_1,,,\lambda_r)(c_j^{(1)},\ldots,\gamma_j^{(1)};C_j)_{1,p_1};\ldots;(c_j^{(r)},\ldots,\gamma_j^{(r)};C^{(r)}{}_j)_{1,p_1} \\ (2\nu+m+n;2\lambda_1,,,2\lambda_r)(b_j;\beta_j^{(1)},\ldots\ldots,\beta_j^{(1)};B_j)_{1,q};(d_j^{(1)},\ldots,\delta_j^{(1`)};D_j^{(1)})_{1,q_1};\ldots;(d_j^{(r)},\ldots,\delta_j^{(r`)};D_j^{(r)})_{1,q_r}\end{array}\right]$$

$$=\sum_{k=0}^{\infty}\left[\frac{(n+k)!}{n!\;k!}sin(n+2k+1)\theta\right] \qquad \times$$

$$I^{0,n:m_1+1,n_1+1;m_r,n_r}_{p,q:p_1+1,q_1;p_r,q_r}\left[\begin{array}{c} z_1 \\ z_r \end{array}\middle|\begin{array}{l}(\nu-k,\lambda_1,,,\lambda_r)(a_j;\alpha_j^{(1)},\ldots\ldots,\alpha_j^{(1)};A_j)_{1,p};(1+\nu+k+n,\ldots,\lambda_1,,,\lambda_r)(c_j^{(1)},\ldots,\gamma_j^{(1)};C_j)_{1,p_1};\ldots;(c_j^{(r)},\ldots,\gamma_j^{(r)};C^{(r)}{}_j)_{1,p_1} \\ (2\nu+n;2\lambda_1,,,2\lambda_r)(b_j;\beta_j^{(1)},\ldots\ldots,\beta_j^{(1)};B_j)_{1,q}(\nu;\lambda_1,,,\lambda_r);(d_j^{(1)},\ldots,\delta_j^{(1`)};D_j^{(1)})_{1,q_1};\ldots;(d_j^{(r)},\ldots,\delta_j^{(r`)};D_j^{(r)})_{1,q_r}\end{array}\right]$$

Proof of main result:

Replacing 'u' by $(1-\nu+\sum_{i=1}^{r}\lambda_i s_i)$, (2.1) takes the form :

$$(sin\theta)^{2\nu-1-2(\sum_{i=1}^{r}\lambda_i s_i)} \times P_n^{(\nu-(\sum_{i=1}^{r}\lambda_i s_i))}(cos\theta)$$

$$=\frac{2^{2\left(1-\nu+\sum_{i=1}^{r}\lambda_i s_i\right)}}{\Gamma\left(\nu-\sum_{i=1}^{r}\lambda_i s_i\right)}\;\frac{\Gamma\left(n+2\nu-2\left(\sum_{i=1}^{r}\lambda_i s_i\right)\right)}{\Gamma\left(1-\nu+\sum_{i=1}^{r}\lambda_i s_i\right)} \quad \times$$

$$\sum_{k=0}^{\infty}\left\{\frac{(n+k)!\,\Gamma\left(k+1-\nu+\sum_{i=1}^{r}\lambda_i s_i\right)}{n!\,k!\,\Gamma\left(n+k+1+\nu-\sum_{i=1}^{r}\lambda_i s_i\right)}\,sin(n+2k+1)\theta\right\}$$

provided $\left(\nu-\sum^{r}\lambda_i s_i\right)>0,\ 0\leq\theta\leq\pi$ and 'n' is a non negative integer.

(3.1)

$$\sqrt{\pi}\frac{1}{(2\pi\omega)^r}\left(\frac{sin\theta}{2}\right)^{2\nu-1}\int_{L_1}\cdots\int_{L_r}\varphi(s_1,\ldots s_r)\prod_{i=1}^{r}\left[\theta_i(s_i)\left(\frac{z_i}{sin^{2\lambda_i}\theta}\right)^{s_i}\right]\times$$

$$\sum_{m=0}^{n}\left[\frac{(cos\theta-1)^m\,\Gamma\left(n+m+2\nu-2(\sum_{i=1}^{r}\lambda_i s_i)\right)}{2^m\,m!\,(n-m)!\,\Gamma\left(\frac{1}{2}+m+\nu-\sum_{i=1}^{r}\lambda_i s_i\right)}\right]ds_1\ldots ds_r$$

(3.2)

$$=\frac{2^{2(1-\nu)}}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\varphi(s_1,..,s_r)\left[\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\right]\frac{\Gamma\left(n+2\nu-2\left(\sum_{i=1}^{r}\lambda_i s_i\right)\right)}{\Gamma\left(1-\nu+\sum_{i=1}^{r}\lambda_i s_i\right)\,n!}\times$$

$$\sum_{k=0}^{\infty}\left[\frac{(n+k)!\,\Gamma\left(1-\nu+k+\sum_{i=1}^{r}\lambda_i s_i\right)}{k!\,\Gamma\left(n+k+\nu+1-\sum_{i=1}^{r}\lambda_i s_i\right)}sin(n+2k+1)\theta\right]ds_1\ldots ds_r$$

(3.3)

Changing the order of integration and summation and using (1.1) the result follows. On the left hand side the change in the order of integration and summation is justified because the series is finite and integrals exist. On the right hand side it is justified because the series converges uniformly with respect to $s_1,s_2,\ldots,s_r$ and $\varphi(s_1,\ldots s_r)$

$\left[\prod_{i=1}^{r}(\theta_i(s_i)z_i^{s_i})\right]\frac{\Gamma\left(n+2\nu-2(\sum_{i=1}^{r}\lambda_i s_i)\right)}{\Gamma\left(1-\nu+\sum_{i=1}^{r}\lambda_i s_i\right)n!}$ is continuous and the integral is absolutely convergent when the given conditions are satisfied.

Conclusion: Here we have established a summation formula for the I- function This result is novel in which we have used multivariable I- function .Due to various applicability of special function these results are very useful.

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