More on $g^{**}\Lambda$- Closed Sets in Topological Spaces

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Abstract: The objective of this paper is to investigate the application on $g^{**}\Lambda$-closed sets in topological spaces. Some fascinating results on $g^{**}\Lambda$-closed sets are obtained by considering T_1 - space, $T_{1/2}$ - space, $T_{1/4}$ - space, ${}^*T_{1/2}$ -space, $T^*_{1/2}$ - space, door space and partition space. Also $g^{**}\Lambda$ - closure of a subset is introduced and studied its properties.

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I Introduction

Levine [6] introduced the notion of generalized closed sets in topological spaces. Veera Kumar [10] defined g^*-closed sets in topological spaces. Maki [7] derived the notion of Λ-sets in topological spaces. A Λ-set is a set A which is equal to its kernel i.e to the intersection of all open supersets of A. Arenas et al.[1] introduced λ-closed sets by using closed sets and Λ -sets. Balamani [2] introduced $g^{**}\Lambda$-closed sets in topological spaces and studied its fundamental properties. In this paper some characterization theorems on $g^{**}\Lambda$-closed sets are analyzed. Further $g^{**}\Lambda$ - closure of a subset is defined and derived its properties.

II Preliminaries

Throughout this paper (X, τ) represents non-empty topological space on which no separation axioms are defined, unless otherwise mentioned. The closure of a subset A of a space (X, τ) is denoted by cl(A).

Definition 2.1

A subset A of a topological space (X, τ) is called a

(1)Regular closed set [9] if A=cl(int(A))

(2) generalized closed set (g-closed) [6] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).

(3) g^*-closed set[10] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in (X, τ).

(4) λ- closed set [1] if $A = C \cap D$ where C is a Λ-set and D is a closed set.

(5) $g\Lambda$-closed set [4] if $cl_\lambda(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).

(6) Λg-closed set [4] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is λ-open in (X, τ).

(7) $g^*\Lambda$-closed set [8] if $cl_\lambda(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in (X, τ).

(8) $g^{**}\Lambda$-closed set [2] if $cl_\lambda(A) \subseteq U$ whenever $A \subseteq U$ and U is g^*-open in (X, τ).

The complements of the above mentioned sets are called their respective open-sets.

(9) The intersection of all λ-closed sets containing A is called the λ - closure of A and is denoted by $cl_\lambda(A)$ [3]

Lemma 2.2

1. In $T_{1/2}$ - space every g - closed subset of (X, τ) is closed. [6]
2. In $T_{1/2}$ - space every subset of (X, τ) is λ- closed. [1]
3. In $T_{1/4}$ - space every finite subset of (X, τ) is λ- closed. [1]
4. In T_1 - space every Λg-closed subset of (X, τ) is closed.[4]
5. In door space every subset of (X, τ) is either open or closed. [5]
6. In $T^*_{1/2}$ -space every g^*-closed subset of (X, τ) is closed. [10]
7. In ${}^*T_{1/2}$ -space every g-closed subset of (X, τ) is g^*-closed.[10]

Remark 2.3

1. Every Λ-set is a λ- closed set [1]
2. Every open and closed sets are λ- closed sets [1]
3. Every λ-closed set is a $g^{**}\Lambda$ - closed set [2]
4. Every open and closed sets are $g^{**}\Lambda$ - closed sets [2]
5. Every $g^{**}\Lambda$-closed set is a $g\Lambda$ - closed set [2]
6. Every $g^*\Lambda$-closed set is a $g^{**}\Lambda$ –closed set [2]

III Application on $g^{**}\Lambda$-Closed Sets

Proposition 3.1 In a T_1 - space, every $g^{**}\Lambda$-closed set is λ- closed

Proof: Let A be a $g^{**}\Lambda$-closed set in T_1 - space. We know that (X, τ) is a T_1 – space if and only if for each $x \in X$, $\{x\}$ is closed. Suppose A is not λ- closed. Then $\exists$ $x \in X$ such that

$\{x\} \in [cl_\lambda(A) - A]$. Since (X, τ) is a T_1 –space, $\{x\}$ is closed. By Theorem 4.1 [2], we get $cl_\lambda(A) - A$ contains no non-empty closed set in X which is a contradiction. Hence A must be λ - closed.

Proposition 3.2 In $T_{1/2}$ -space every subset of (X, τ) is $g**\Lambda$-closed.

Proof: Follows from Lemma 2.2 (2) and by Remark 2.3(3)

Proposition 3.3 Every finite subset of $T_{1/4}$ -space is $g**\Lambda$-closed.

Proof: Follows from Lemma 2.2(3) and by Remark 2.3(3)

Proposition 3.4 In a door space every subset is $g**\Lambda$-closed but not conversely.

Proof: Follows from Lemma 2.2(5) and by Remark 2.3(4)

Example 3.5 Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}\}$. Then every subset is $g**\Lambda$-closed but it is not a door space.

Proposition 3.6 In general Λg-closed sets and $g**\Lambda$-closed sets are independent, but in T_1 - space every Λg-closed set is $g**\Lambda$-closed.

Proof: Follows from Lemma 2.2(4) and by Remark 2.3(4)

Proposition 3.7 In general g-closed sets and $g**\Lambda$-closed are independent but in $T_{1/2}$ - space every g-closed set is $g**\Lambda$-closed.

Proof: Follows from Lemma 2.2(1) and by Remark 2.3(4)

Proposition 3.8 In ${}^*T_{1/2}$ -space every $g**\Lambda$-closed set in (X, τ) is $g*\Lambda$ -closed.

Proof: Let A be a $g**\Lambda$- closed set of (X, τ). Let U be any g -open set containing A in X. Since (X, τ) is a ${}^*T_{1/2}$ -space, every g-open set is $g*$-open and A is $g**\Lambda$-closed set, $cl_\lambda(A) \subseteq U$. Hence A is $g*\Lambda$ - closed.

Theorem 3.9 Let (X, τ) be a topological space. Then the following statements are equivalent.

(i) Every $g*$-open set is λ- closed

(ii) Every subset is $g**\Lambda$-closed.

Proof: (i) $\Rightarrow$ (ii)

Let A be any subset of (X, τ) such that $A \subseteq U$ and U is $g*$ -open. Hence we get $cl_\lambda(A) \subseteq cl_\lambda(U))$. By assumption U is λ- closed. Hence $cl_\lambda(A) \subseteq cl_\lambda(U)) = U$. Therefore A is $g**\Lambda$-closed.

(ii) $\Rightarrow$ (i) Let A be a $g*$-open set. By assumption A is $g**\Lambda$-closed. Hence we get $cl_\lambda(A) \subseteq A$.

Therefore A is λ- closed.

Theorem 3.10 Let A be a subset of $T^*_{1/2}$ -space. Then the following statements are equivalent.

(i) A is a $g^{**}\Lambda$-closed set

(ii) A is a $g\Lambda$-closed set

Proof: (i) $\Rightarrow$ (ii) Follows from Remark 2.3(5)

(ii) $\Rightarrow$ (i) Let A be a $g\Lambda$-closed set. Let $A \subseteq U$ where U is g^*-open. Since (X, τ) is a $T^*_{1/2}$ -space, U is open and A is $g\Lambda$-closed, $cl_\lambda(A) \subseteq U$. Therefore A is $g^{**}\Lambda$-closed.

Theorem 3.11 Let A be a subset of ${}^*T_{1/2}$ -space. Then the following statements are equivalent.

(i) A is a $g^*\Lambda$-closed set

(ii) A is a $g^{**}\Lambda$ -closed set

Proof: (i) $\Rightarrow$ (ii) Follows from Remark 2.3(6)

(ii) $\Rightarrow$ (i) Let A be a $g^{**}\Lambda$-closed set. Let $A \subseteq U$ where U is g-open. Since (X, τ) is a ${}^*T_{1/2}$ -space, U is g^*-open and A is $g^{**}\Lambda$-closed , $cl_\lambda(A) \subseteq U$. Therefore A is $g^*\Lambda$-closed.

Proposition 3.12 Let A be a $g^{**}\Lambda$-closed set of a topological space (X, τ). Then

(i) If A is regular open then scl(A) and pint(A) are also $g^{**}\Lambda$-closed.

(ii) If A is regular closed then pcl(A) and sint(A) are also $g^{**}\Lambda$**-**closed.

Proof: (i) Since A is regular open in (X, τ), A= int(cl(A)). By definition scl(A) = A $\cup$ int(cl(A)) = A. Therefore scl(A) is $g^{**}\Lambda$-closed in (X, τ). By definition pint(A) = A $\cap$ int(cl(A)) = A. Hence pint(A) is $g^{**}\Lambda$-closed.

(ii) Since A is regular closed in (X, τ), A= cl(int(A)). By definition pcl(A) = A $\cup$ cl(int(A)) = A. Therefore pcl(A) is $g^{**}\Lambda$-closed in (X, τ). By definition sint(A) = A $\cap$ cl(int(A)) = A. Hence sint(A) is $g^{**}\Lambda$-closed.

Definition 3.13 [3] Let A be a subset of a space (X, τ). A point x $\in$ X is said to be a λ - limit point of A if for every λ - open set containing x, $U \cap [A \setminus \{x\}] \neq \varphi$. The set of all λ- limit points of A is said to be the λ - derived set of A and is denoted by $D_\lambda(A)$.

Remark 3.14 For any subset A of a space (X, τ) the following properties hold

- $D_\lambda(A) \subseteq D(A)$ where $D(A)$ is the derived set of A.
- $cl_\lambda(A) = A \cup D_\lambda(A)$

Theorem 3.15 If A and B are $g^{**}\Lambda$- closed sets such that $D(A) \subseteq D_\lambda(A)$ and $D(B) \subseteq D_\lambda(B)$, then $A \cup B$ is a $g^{**}\Lambda$- closed set.

Proof: From hypothesis and by Remark 3.14, we get $D(A) = D_\lambda(A)$ and $D(B) = D_\lambda(B)$). Therefore we have $cl(A) = A \cup D(A) = A \cup D_\lambda(A) = cl_\lambda(A)$ Similarly, $cl(B) = cl_\lambda(B)$. Let U be a g^* - open set such that $A \cup B \subseteq U$. Since A and B are $g^{**}\Lambda$- closed sets, $cl_\lambda(A) \subseteq U$ and $cl_\lambda(B) \subseteq U$. Thus $cl_\lambda(A \cup B) \subseteq cl(A \cup B) = cl(A) \cup cl(B) = cl_\lambda(A) \cup cl_\lambda(B) \subseteq U$, which implies that $A \cup B$ is $g^{**}\Lambda$- closed.

Definition 3.16 A partition space is a space where every open set is closed.

Theorem 3.17 Let (X, τ) be a $T^*_{1/2}$ -space. If X is a partition space, then every subset of X is a $g^{**}\Lambda$-closed set.

Proof: Let A be any subset of (X, τ) such that $A \subseteq U$ and U is g^* -open. Since (X, τ) is a $T^*_{1/2}$ -space, U is open. Since X is a partition space, U is closed. Since every closed set is λ- closed, U is λ-closed. Hence $cl_\lambda(A) \subseteq cl_\lambda(U) = U$ Therefore every subset of X is $g^{**}\Lambda$-closed.

Theorem 3.18 In a partition space every $g^{**}\Lambda$- closed set is a g-closed set.

Proof: Let A be a $g^{**}\Lambda$- closed set of (X ,τ). Let U be any open set containing A in X. Since every open set is g^*-open and A is $g^{**}\Lambda$-closed, $cl_\lambda(A) \subseteq U$. Since (X, τ) is a partition space, closed sets coincide with λ- closed sets. Hence we have $cl(A) = cl_\lambda(A) \subseteq U$. Therefore A is a g-closed set.

Theorem 3.19 If A is a $g^{**}\Lambda$- closed set, then $g^*cl(\{x\}) \cap A \neq \varphi$ for every $x \in cl_\lambda(A)$.

Proof: Let A be a $g^{**}\Lambda$ - closed set of (X,τ). Suppose $g^{*}cl(\{x\}) \cap A = \varphi$ for some $x \in cl_{\lambda}(A)$.

Then (X- $g^{*}cl(\{x\})$ is a g^{*}-open set containing A . Also $x \in cl_{\lambda}(A)$ and $x \notin$ (X- $g^{*}cl(\{x\})$ which implies $x \in [cl_{\lambda}(A)$ - (X- $g^{*}cl(\{x\})]$. Therefore $cl_{\lambda}(A)$ is not contained in (X- $g^{*}cl(\{x\})]$ is a contradiction to A is a $g^{**}\Lambda$ - closed set. Hence $g^{*}cl(\{x\}) \cap A \neq \varphi$ for every $x \in cl_{\lambda}(A)$.

IV $g^{**}\Lambda$ - Closure

Definition 4.1 The $g^{**}\Lambda$ -closure of a subset A of a topological space (X, τ) denoted by $g^{**}\Lambda cl(A)$) is defined as follows.

$g^{**}\Lambda cl(A) = \cap \{F \subseteq X: A \subseteq F$ and F is $g^{**}\Lambda$ - closed in $(X, \tau)\}$

Remark 4.2 For a subset A of a topological space (X, τ), $A \subseteq g^{**}\Lambda cl(A) \subseteq cl_{\lambda}(A) \subseteq cl(A)$

Proof: Follows from Remark 2.3(2)(3) and Definition 4.1.

Proposition 4.3 If A is a $g^{**}\Lambda$ -closed set in (X, τ), then $g^{**}\Lambda cl(A) = A$.

Proof : Let A be a $g^{**}\Lambda$ -closed set in (X, τ). Then by the definition of $g^{**}\Lambda cl(A)$, the smallest F in the intersection of $g^{**}\Lambda$ -closed sets containing A is A itself. Hence $g^{**}\Lambda cl(A) = A$.

Remark 4.4 Since the intersection of two $g^{**}\Lambda$ -closed sets need not be $g^{**}\Lambda$ -closed, $g^{**}\Lambda cl(A)$ may not be a $g^{**}\Lambda$ -closed set.

Example 4.5 Let X = {a, b, c} with τ = { X, ϕ, {a, b}}. Then $g^{**}\Lambda$ -closed sets in (X, τ) are X, ϕ, {c}, {a, b}, {a, c}, {b, c}.Here $g^{**}\Lambda cl(\{a\}) = \cap \{X, \{a, b\}, \{a, c\}\} = \{a\}$ which is not a $g^{**}\Lambda$ -closed set in (X, τ).

Proposition 4.6 Let A and B be any two subsets of (X, τ). If $A \subseteq B$, then $g^{**}\Lambda cl(A) \subseteq g^{**}\Lambda cl(B)$.

Proof: Follows from the Definition 4.1.

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