Mapping Properties of Generalized hypergeometric distribution series on certain classes of univalent functions

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Abstract

The purpose of the present paper is to obtain some necessary and sufficient conditions for generalized hypergeometric distribution series belonging to the classes $T(\lambda,\alpha), C(\lambda,\alpha), S^*_\delta$ and C_δ. Further, we obtain some inclusion relation between the classes $R^\tau(A,B)$ with C_δ and S^*_δ. Finally, we discuss an integral operator associated with generalized hypergeometric distribution series.

Keywords: Analytic, univalent, generalized hypergeometric series.
AMS 2010 Mathematics Subject Classification: 30C45.

1 Introduction

Let A represent the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. As usual, we denote by S the subclass of A consisting of functions which are univalent in U.

We further denote by T be the subclass of S consisting the functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n. \tag{1.2}$$

A function f of the form (1.2) is said to be in the class $T(\lambda, \alpha)$ if it satisfy the following condition

$$\Re\left\{\frac{zf'(z)}{\lambda zf'(z)+(1-\lambda)f(z)}\right\} > \alpha, \qquad z \in U, \tag{1.3}$$

where α and λ are non-negative real numbers with $0 \le \alpha < 1$ and $0 \le \lambda \le 1$. Similarly, a function f of the form (1.2) is said be in the class $C(\lambda, \alpha)$ if it satisfy the following condition

$$\Re\left\{\frac{f'(z)+zf''(z)}{f'(z)+\lambda zf''(z)}\right\} > \alpha, \qquad z \in U, \tag{1.4}$$

where α and λ are non-negative real numbers with $0 \le \alpha < 1$ and $0 \le \lambda \le 1$.

From the conditions (1.3) and (1.4), we have

$$f(z) \in C(\lambda, \alpha) \Longleftrightarrow zf'(z) \in T(\lambda, \alpha).$$

The classes $T(\lambda, \alpha)$ and $C(\lambda, \alpha)$ were extensively studied by Altintas and Owa [2].

In 1998, Ponnusamy and Rønning [10] introduced the class S^*_δ and C_δ in the following way

A function $f(z)$ of the form (1.1) is said to be in the class S^*_δ if it satisfy the following condition

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \delta, \tag{1.5}$$

where $\delta > 0$.

A function $f(z)$ of the form (1.1) is said to be in the class C_δ if it satisfy the following condition

$$\left|\frac{zf''(z)}{f'(z)}\right| < \delta, \tag{1.6}$$

where $\delta > 0$.

From condition (1.5) and (1.6) one can easy to verify that

$$f(z) \in C_\delta \Longleftrightarrow zf'(z) \in S^*_\delta.$$

In 1995, Dixit and Pal [7] introduce the class $R^\tau(A, B)$ consisting of functions $f(z)$ of the form (1.1) which satisfy the condition

$$\left|\frac{f'(z)-1}{(A-B)\tau - B(f'(z)-1)}\right| < 1,$$

where $\tau \in C\backslash\{0\}, -1 \le B < A \le 1,\ z \in U$.

The applications of hypergeometric function [10, 11] Generalized hypergeometric function [9], Bessel function [14] and generalized Bessel functions [5] are interesting topic of research in geometric function theory. In 2014, Porwal [12] introduce Poisson distribution series and gave a nice application of it on certain classes of univalent functions. They obtain some sufficient conditions for Poisson distribution series belonging to certain classes of univalent functions. The result of this paper are important from this point of view that they depend only one parameter. After the appearance of this paper several researchers introduce confluent hypergeometic distribution series [15], hypergeometric distribution series [1], hypergeometric type distribution series [16], Pascal distribution series [6], Borel distribution series [19], generalized distribution series [13], generalized hypergeometric distribution series [18] and Mittag-Leffler type Poisson distribution series [4] and obtain several interesting results on various subclasses of univalent functions. In this paper, motivating with the above mentioned work, we obtain some necessary and sufficient conditions for generalized hypergeometric distribution series belonging to the classes $T(\lambda, \alpha)$, $C(\lambda, \alpha)$, S^*_δ and C_δ.

Very recently, Themangani et al. [18] introduce generalized hypergeometric distribution series which is a natural generalization of confluent hypergeometric distribution series, hypergeometric type distribution series and Poisson distribution series.

Now, we recall the definition of generalized hypergeometric function which may be found in [17].

For $a_1, a_2, ..., a_p$ and $b_1, b_2, ..., b_q$ be the complex numbers with $b_j \neq 0, -1, -2, ...$ $j = 1, 2, ..., q$ then the generalized hypergeometric function ${}_pF_q(a_1, a_2, ..., a_p; b_1, b_2, ...b_q)$ is defined by

$$ {}_pF_q(a_1, a_2, ..., a_q;\ b_1, b_2, ..., b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n ...(a_p)_n}{(b_1)_n ...(b_q)_n} \frac{z^n}{n!}, \qquad z \in U, \quad (1.7)$$

where $p \leq q + 1$ and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0 \\ a(a+1)...(a+n-1), & \text{if } n \in N \end{cases}.$$

The convergence condition of the series defined by (1.7) is given below

1. If $p < q + 1$ then the series converges absolutely in the entire complex plane.
2. If $p \leq q$ then the series converges absolutely for every finite z.
3. If $p = q + 1$ then the series converges absolutely when $|z| < 1$.
4. If $p = q + 1$ and $|z| = 1$ then the series converges when $\Re\left\{\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i\right\} > 0$.

Now, we define for $a_i, b_j, m > 0$, where $i = 1, 2, ..., p$ and $j = 1, 2, ..., q$ such that the series

$$ {}_pF_q(a_1, ..., a_p; b_1, ..., b_q; m) = \sum_{n=0}^{\infty} \frac{(a_1)_n ...(a_p)_n}{(b_1)_n ...(b_q)_n} \frac{z^n}{n!}$$

is convergent.

Throughout this article, we will frequently use the notation

$$ {}_pF_q(z) = {}_p F_q(a_1,\ a_2,\ \cdots a_p;\ b_1,\ b_2,\ \cdots b_q; z) $$

and

$$ {}_pF_q(a_1{+}k;\ b_1{+}k;\ z) = {}_p F_q(a_1{+}k,\ a_2{+}k,\ \cdots a_p{+}k;\ b_1{+}k,\ b_2{+}k,\ \cdots b_q{+}k; z),\quad k \in \mathbb{N}. $$

Next, Themangani et al. [18] introduce the generalized hypergeometric distribution whose probability mass function is

$$ \frac{(a_1)_n ...(a_p)_n}{(b_1)_n ...(b_q)_n} \frac{m^n}{{}_pF_q(m)n!}, \qquad n = 0, 1, 2, \ldots $$

By specializing the parameters in the generalized hypergeometric distribution it reduce to the following probability distribution

(1) If $p = 2,\ q = 1$ then it reduces to the hypergeometric type probability distribution studied by Porwal and Gupta [16].

(2) If $p = q = 1$ then it reduces to the confluent hypergeometric distribution studied by Porwal and Kumar [15].

(3) If $p = q = 1$ and $a_1 = b_1$, then it reduces to the well-known Poisson distribution.

Next, Themangani et al. [18] introduce the generalized hypergeometric distribution series whose coefficients are probabilities of generalized hypergeometric distribution

$$ {}_pF_q(m, z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1} ...(a_p)_{n-1}}{(b_1)_{n-1} ...(b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \frac{z^n}{{}_pF_q(m)}, \tag{1.8} $$

where $a_i,\ b_j > 0,\ i = 1, 2, ..., p;\ j = 1, 2, ..., q$.

Next, we define

$$ \overline{{}_pF_q}(m, z) = 2z - {}_pF_q(m, z) $$

$$ \overline{{}_pF_q}(m, z) = z - \sum_{n=2}^{\infty} \frac{(a_1)_{n-1} ...(a_p)_{n-1}}{(b_1)_{n-1} ...(b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \frac{z^n}{{}_pF_q(m)}. \tag{1.9} $$

The convolution (or, Hadamard product) of two power series is defined as

$$ (f * g)(z) = f(z) * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \quad z \in U. $$

Next, we consider a linear operator $\Omega(p, q, m) :\ A \to A$ defined by

$$ \Omega(p, q, m)f(z) = {}_pF_q(m, z) * f(z) $$

$$\Omega(p,q,m)f(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \frac{a_n z^n}{{}_pF_q(m)}.$$

In the present paper, we obtain necessary and sufficient conditions for the generalized hypergeometric distribution series belonging to the classes $T(\lambda,\alpha)$, $C(\lambda,\alpha)$, S_*^δ and C_δ. We also discuss an integral operator associated with the generalized hypergeometric distribution series.

2 Preliminary Result

To prove our main results we shall require the following lemmas.

Lemma 2.1. ([2]) A function f defined by (1.2) is in the class $T(\lambda,\alpha)$ if and only if $\sum_{n=2}^{\infty}\{n(1-\lambda\alpha)-\alpha(1-\lambda)\}|a_n| \le 1-\alpha$.

Lemma 2.2. ([2]) A function f defined by (1.2) is in the class $C(\lambda,\alpha)$, if and only if $\sum_{n=2}^{\infty} n\{n(1-\lambda\alpha)-\alpha(1-\lambda)\}|a_n| \le 1-\alpha$.

Lemma 2.3. ([10]) A function f defined by (1.1) and satisfy the condition $\sum_{n=2}^{\infty}(n+\delta-1)|a_n| \le \delta$, then $f \in S_\delta^*$.

Lemma 2.4. ([10]) A function f defined by (1.1) and satisfy the condition $\sum_{n=2}^{\infty} n(n+\delta-1)|a_n| \le \delta$, then $f \in C_\delta$.

Lemma 2.5. ([7]) If $f \in R^\tau(A,\ B)$ is of the form (1.1) then

$$|a_n| \le \frac{(A-B)\,|\tau|}{n}, \qquad n \in \mathbb{N}\setminus\{1\}. \tag{2.1}$$

The bounds given in (2.1) is sharp.

3 Main Results

In our first theorem we obtain a necessary and sufficient condition for the generalized hypergeometric distribution series $\overline{{}_pF_q}(m,z)$ belonging to the class $T(\lambda,\alpha)$.

Theorem 3.1. If $a_i, b_j > 0 \quad (i=1,2,...,p;\ j=1,2,...,q)$ then $\overline{{}_pF_q}(m,z)$ is in the class $T(\lambda,\alpha)$, if and only if

$$(1-\alpha\lambda)\frac{a_1...a_p}{b_1...b_q} m_pF_q(a_1+1;\ b_1+1;m) \le 1-\alpha, \tag{3.1}$$

holds with either one of the conditions
(i) $p \leq q$ and $m > 0$
(ii) $p = q+1$ and $m < 1$
(iii) $p = q+1,\ m = 1$ and $\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 1$

Proof. To prove $\overline{{}_pF_q}(m,z) \in T(\lambda,\alpha)$, it is sufficient to prove that

$$\sum_{n=2}^{\infty}\{n(1-\alpha\lambda)-\alpha(1-\lambda)\}\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{(n-1)!}\frac{1}{{}_pF_q(m)} \leq 1-\alpha.$$

Now

$$\sum_{n=2}^{\infty}\{n(1-\alpha\lambda)-\alpha(1-\lambda)\}\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{(n-1)!}\frac{1}{{}_pF_q(m)}$$

$$= \sum_{n=2}^{\infty}\{(1-\alpha\lambda)(n-1)+1-\alpha\}\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{(n-1)!}\frac{1}{{}_pF_q(m)}$$

$$= \frac{1}{{}_pF_q(m)}\left[\sum_{n=2}^{\infty}(1-\alpha\lambda)\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{(n-2)!}\right.$$

$$\left.+(1-\alpha)\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{(n-1)!}\right]$$

$$= \frac{1}{{}_pF_q(m)}\left[(1-\alpha\lambda)\frac{(a_1)...(a_p)}{(b_1)...(b_q)}m\sum_{n=2}^{\infty}\frac{(a_1+1)_{n-2}...(a_p+1)_{n-2}}{(b_1+1)_{n-2}...(b_q+1)_{n-2}}\frac{m^{n-2}}{(n-2)!}\right.$$

$$\left.+(1-\alpha)\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{(n-1)!}\right]$$

$$= \frac{1}{{}_pF_q(m)}\left[(1-\alpha\lambda)\frac{(a_1)...(a_p)m}{(b_1)...(b_q)}{}_pF_q(a_1+1;\ b_1+1;\ m)\right.$$

$$\left.+(1-\alpha)({}_pF_q(m)-1)\right]$$

$$\leq 1-\alpha,$$

by the given hypothesis.

This completes the proof of Theorem 3.1. □

Theorem 3.2. If $a_i, b_j > 0 (i = 1,2,...,p; j = 1,2,...,q)$ then $\overline{{}_pF_q}(m,z)$ is in the class $C(\lambda,\alpha)$, if and only if

$$(1-\alpha\lambda)\frac{a_1(a_1+1)...a_p(a_p+1)m^2}{b_1(b_1+1)...b_q(b_q+1)}$$

$$ {}_pF_q(a_1+2;\ b_1+2; m)+(3-2\alpha\lambda-\alpha)\frac{a_1...a_p}{b_1...b_q}m\,{}_pF_q(a_1+1;\ b_1+1;\ m) \leq 1-\alpha \quad (3.2) $$

holds with either one of the conditions
(i) $p < q$ and $m > 0$
(ii) $p = q + 1$ and $m < 1$
(iii) $p = q + 1,\ m = 1$ and $\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 2$

Proof. To prove that $\overline{{}_pF_q}(m, z) \in C(\lambda, \alpha)$ it is sufficient to prove that

$$ \sum_{n=2}^{\infty} n\{n(1-\alpha\lambda)-\alpha(1-\lambda)\}\frac{(a_1)_{n-1}...(a_p)_{n-1}m^{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}(n-1)!}\frac{1}{{}_pF_q(m)} \leq 1-\alpha. $$

Now

$$ \sum_{n=2}^{\infty} n\{n(1-\alpha\lambda)-\alpha(1-\lambda)\}\frac{(a_1)_{n-1}...(a_p)_{n-1}m^{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}(n-1)!}\frac{1}{{}_pF_q(m)} $$

$$ = \frac{1}{{}_pF_q(m)}\sum_{n=2}^{\infty} n\{n(1-\alpha\lambda)-\alpha(1-\lambda)\}\frac{(a_1)_{n-1}...(a_p)_{n-1}m^{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}(n-1)!} $$

$$ = \frac{1}{{}_pF_q(m)}\left[\sum_{n=2}^{\infty}\{(1-\alpha\lambda)(n-1)(n-2)+(3-2\alpha\lambda-\alpha)(n-1)\right. $$

$$ \left. +1-\alpha\}\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{(n-1)!}\right] $$

$$ = \frac{1}{{}_pF_q(m)}\left[(1-\alpha\lambda)\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{(n-3)!}\right. $$

$$ +(3-2\alpha\lambda-\alpha)\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{(n-2)!} $$

$$ \left. +(1-\alpha)\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{(n-1)!}\right] $$

$$ = \frac{1}{{}_pF_q(m)}\left[(1-\alpha\lambda)\frac{a_1(a_1+1)...a_p(a_p+1)}{b_1(b_1+1)...b_1(b_q+1)}m^2\right. $$

$$ \times\sum_{n=2}^{\infty}\frac{(a_1+2)_{n-3}...(a_p+2)_{n-3}}{(b_1+2)_{n-3}...(b_q+2)_{n-3}}\frac{m^{n-3}}{(n-3)!}+(3-2\alpha\lambda-\alpha)\frac{a_1...a_pm}{b_1...b_q} $$

$$ \left. \times\sum_{n=2}^{\infty}\frac{(a_1+1)_{n-2}...(a_p+1)_{n-2}m^{n-2}}{(b_1+1)_{n-2}...(b_q+1)_{n-2}(n-2)!}+(1-\alpha)({}_pF_q(m)-1)\right] $$

$$= \frac{1}{{}_pF_q(m)}\left[(1-\alpha\lambda)\frac{a_1(a_1+1)...a_p(a_p+1)}{b_1(b_1+1)...b_1(b_q+1)}m^2{}_pF_q(a_1+2;\ b_1+2;\ m)\right.$$

$$\left.+(3-2\alpha\lambda-\alpha)\frac{a_1...a_pm}{b_q...b_1}{}_pF_q(a_1+1;\ b_1+1;\ m)+(1-\alpha)({}_pF_q(m)-1)\right] \leq 1-\alpha$$

by given hypothesis.

This completes the proof of Theorem 3.2. □

Theorem 3.3. If $a_i,\ b_j > 0(i = 1, 2, ..., p;\ j = 1, 2, ...q)$ and the inequality

$$\frac{a_1...a_pm}{b_1...b_q}{}_pF_q(a_1+1;\ b_1+1;\ m) \leq \delta \tag{3.3}$$

holds with either one of the conditions
(i) $p \leq q$ and $m > 0$
(ii) $p = q + 1$ and $m < 1$
(iii) $p = q + 1,\ m = 1$ and $\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 1$
then ${}_pF_q(m, z)$ defined by (1.8) is in the class S^*_δ.

Proof. To prove that ${}_pF_q(m, z)$ defined by (1.8) is in the class S^*_δ, from Lemma 2.3, it is sufficient to show that

$$\sum_{n=2}^{\infty}(n+\delta-1)\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{(n-1)!}\frac{1}{{}_pF_q(m)} \leq \delta.$$

Now

$$\sum_{n=2}^{\infty}(n+\delta-1)\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{(n-1)!}\frac{1}{{}_pF_q(m)}$$
$$= \frac{1}{{}_pF_q(m)}\left[\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{(n-2)!} + \delta\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{(n-1)!}\right]$$
$$= \frac{1}{{}_pF_q(m)}\left[\frac{a_1...a_pm}{b_1...b_q}\sum_{n=2}^{\infty}\frac{(a_1+1)_{n-2}...(a_p+1)_{n-2}}{(b_1+1)_{n-2}...(b_q+1)_{n-2}}\frac{m^{n-2}}{(n-2)!} + \delta({}_pF_q(m)-1)\right] \leq \delta, \tag{3.4}$$

by given hypothesis.

This completes the proof of Theorem 3.3. □

Theorem 3.4. If $a_i,\ b_j > 0\ (i = 1, 2, ..., p;\ j = 1, 2, ..., q)$ and the inequality

$$\frac{a_1(a_1+1)...a_p(a_p+1)m^2}{b_1(b_1+1)...b_q(b_q+1)}{}_pF_q(a_1+2;\ b_q+2;\ m)+(\delta+2)\frac{a_1...a_p}{b_1...b_q}m\,{}_pF_q(a_1+1;\ b_1+1;\ m) \leq \delta \tag{3.5}$$

holds with either one of the conditions
(i) $p \leq q$ and $m > 0$

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(ii) $p = q + 1$ and $m < 1$
(iii) $p = q + 1,\ m = 1$ and $\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 2$ then ${}_pF_q(m, z)$ defined by (1.8) is in the C_δ.

Proof. The proof of above theorem is much similar to that of Theorem 3.3 therefore, we omit the details involved. □

4 Inclusion Relation

In this section we obtain inclusion relation between the classes $R^\tau(A, B)$ and $S_\delta^*,\ C_\delta$.

Theorem 4.1. If $a_i, b_j > 0 \quad (i = 1, 2, ..., p;\ j = 1, 2, ..., q),\ f \in R^\tau(A, B)$ and the inequality

$$\frac{(A-B)|\tau|}{{}_pF_q(m)}\left[\frac{a_1...a_p}{b_1...b_q} m\ {}_pF_q(a_1+1;\ b_1+1;\ m) + \delta({}_pF_q(m) - 1)\right] \leq \delta$$

holds with either one of the conditions
(i) $p \leq q$ and $m > 0$
(ii) $p = q + 1$ and $m < 1$
(iii) $p = q + 1,\ m = 1$ and $\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 1$
then $\Omega(p, q;\ m)f \in C_\delta$.

Proof. Since

$$\Omega(p, q, m)f(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \frac{a_n z^n}{{}_pF_q(m)}.$$

To prove that $\Omega(p, q, m)f \in C_\delta$, from Lemma 2.4, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n(n+\delta-1) \frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \frac{|a_n|}{{}_pF_q(m)} \leq 1 - \alpha.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n+\delta-1) \frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \frac{|a_n|}{{}_pF_q(m)} \\
&= \frac{(A-B)|\tau|}{{}_pF_q(m)} \sum_{n=2}^{\infty} (n+\delta-1) \frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \\
&= \frac{(A-B)|\tau|}{{}_pF_q(m)} \left\{\delta \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}} \frac{m^{n-1}}{(n-2)!}\right\} \\
&= \frac{(A-B)|\tau|}{{}_pF_q(m)} \left\{\delta({}_pF_q(m) - 1) + \frac{a_1...a_p}{b_1...b_q} m\ {}_pF_q(a_1+1;\ b_1+1;\ m)\right\} \leq \delta,
\end{aligned} \tag{4.1}$$

by the given hypothesis.

This completes the proof of Theorem 4.1. □

Theorem 4.2. If $a_i,\ b_j > 1\ (i = 1, 2, ..., p;\ j = 1, 2, ..., q),\ f \in R^{\tau}(A, B)$ and the inequality

$$\frac{(A-B)|\tau|}{{}_pF_q(m)}\Bigg[{}_pF_q(m)-1+(\delta-1)\Bigg\{\frac{(b_1-1)...(b_1-1)}{(a_1-1)...(a_q-1)m}{}_pF_q(a_1-1;\ b_1-1;\ m)$$
$$-\frac{b_1-1)...(b_1-1)}{(a_1-1)...(a_1-1)m}-1\Bigg\}\Bigg]\leq\delta$$

holds with either one of the conditions

(i) $p \leq 1$ and $m > 0$

(ii) $p = q + 1$ and $m < 1$

(iii) $p = q + 1,\ m = 1$ and $\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i$

then $\Omega(p, q, m)f \in S^{*}_{\delta}$.

Proof. The proof of above theorem is much akin to that of Theorem 4.1. Hence, we omit the details involved. □

5 An integral operator

In this section, we obtain analogues results for a particular integral operator ${}_pI_q(m, z)$ defined as

$${}_pI_q(m,z)=\int_0^z\frac{{}_pF_q(m,t)}{t}dt \tag{5.1}$$

Theorem 5.1. If $a_i,\ b_j > 0(i = 1, 2, ..., p;\ j = 1, 2, ..., q)$ and the inequality (3.3) is satisfied then ${}_pI_q(m, z) \in C_{\delta}$.

Proof. From the representation of (5.1) we have

$${}_pI_q(m,z)=z+\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{{}_pF_q(m)}\frac{z^n}{n!}.$$

To prove that ${}_pI_q(m, z) \in C_{\delta}$, from Lemma 2.4, it is sufficient to show that

$$\sum_{n=2}^{\infty}n(n+\delta-1)\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{(n)!}\frac{1}{{}_pF_q(m)}\leq\delta.$$

Now

$$\sum_{n=2}^{\infty}n(n+\delta-1)\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{(n)!}\frac{1}{{}_pF_q(m)}$$
$$=\sum_{n=2}^{\infty}(n+\delta-1)\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{(n-1)!}\frac{1}{{}_pF_q(m)}$$

$$= \frac{1}{{}_pF_q(m)}\left\{\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{(n-2)!}+\delta\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{(n-1)!}\right\}$$

$$= \frac{1}{{}_pF_q(m)}\left\{\frac{a_1...a_p}{b_1...b_q}m\ {}_pF_q(a_1+1;\ b_1+1;\ m)+\delta({}_pF_q(m)-1)\right\}\leq\delta,\quad \text{by the given hypothesis}$$

This completes the proof of Theorem 5.1. □

Theorem 5.2. If $a_i,\ b_j > 1$ $(i = 1, 2, ..., p;\ j = 1, 2, ..., q)$ and the inequality

$$(\delta-1)\frac{(b_1-1)...(b_q-1)}{(a_1-1)...(a_p-1)m}\left\{{}_pF_q(a_1-1;\ b_1-1;\ m)-1\right\}$$

$$\leq(\delta-1){}_pF_q(m)+\delta$$

is satisfied then ${}_pI_q(m,z)\in S^*_\delta$.

Proof. To prove that ${}_pI_q(m,z)\in S^*_\delta$, from Lemma 2.3, it is sufficient to show that

$$\sum_{n=2}^{\infty}(n+\delta-1)\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{1}{n!}\frac{m^{n-1}}{{}_pF_q(m)}\leq\delta.$$

Now

$$\sum_{n=2}^{\infty}(n+\delta-1)\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{n!}\frac{1}{{}_pF_q(m)}$$

$$= \frac{1}{{}_pF_q(m)}\left\{\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{(n-1)!}+(\delta-1)\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}}\frac{m^{n-1}}{n!}\right\}$$

$$= \frac{1}{{}_pF_q(m)}\left\{{}_pF_q(m)-1+(\delta-1)\frac{(b_1-1)...(b_q-1)}{(a_1-1)...(a_p-1)m}\sum_{n=2}^{\infty}\frac{(a_1-1)_n...(a_p-1)_n}{(b_1-1)_n...(b_q-1)_n}\frac{m^n}{n!}\right\}$$

$$= \frac{1}{{}_pF_q(m)}\left\{{}_pF_q(m)-1+(\delta-1)\frac{(b_1-1)...(b_q-1)}{(a_1-1)...(a_p-1)m}\left({}_pF_q(a_1-1;\ b_1-1;\ m)-1-\frac{(a_1-1)...(a_p-1)m}{(b_1-1)...(b_q-1)}\right)\right\}$$

$$\leq\delta$$

by given hypothesis.

This completes the poof of the Theorem 5.2. □

Theorem 5.3. If $a_i,\ b_j > 0$ $(i = 1, 2, ..., p;\ j = 1, ..., q)$ then ${}_p\bar{I}_q(m,z) = 2z - {}_pI_q(m,z)$ is in the class $C(\lambda,\alpha)$, if and only if the condition (3.1) holds with either one of the conditions

1. $p \leq q$ and $m > 0$
2. $p = q + 1$ and $m < 1$
3. $p = q + 1,\ m = 1$ and $\sum_{j=1}^{q} b_j > \sum_{i=q}^{b} a_i + 1.$

Proof. Since

$$
\begin{aligned}
&{}_p\bar{I}_q(m,z) = 2z - {}_pI_q(m,z)\\
&\Rightarrow {}_p\bar{I}_q(m,z) = z - \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}} \frac{m^{n-1}}{n!} \frac{z^n}{{}_pF_q(m)}.
\end{aligned}
$$

To prove that ${}_p\bar{I}_q(m,z) \in C(\lambda,\alpha)$, from Lemma 2.2, it is sufficient to prove that

$$
\begin{aligned}
&\sum_{n=2}^{\infty} n\{n(1-\alpha\lambda) - \alpha(1-\lambda)\} \frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}} \frac{m^{n-1}}{(n)!} \frac{1}{{}_pF_q(m)}\\
&= \frac{1}{{}_pF_q(m)} \sum_{n=2}^{\infty} \{n(1-\alpha\lambda) - \alpha(1-\lambda)\} \frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!}\\
&= \frac{1}{{}_pF_q(m)} \sum_{n=2}^{\infty} \{(1-\alpha\lambda)(n-1) + 1 - \alpha\} \frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!}\\
&= \frac{1}{{}_pF_q(m)} \left[(1-\alpha\lambda) \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}} \frac{m^{n-1}}{(n-2)!}\right.\\
&\left. +(1-\alpha) \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!}\right]\\
&= \frac{1}{{}_pF_q(m)} \left[(1-\alpha\lambda) \frac{a_1...a_p m}{b_1...b_q} {}_pF_q(a_1+1;\ b_1+1,\ m)\right.\\
&\left. +(1-\alpha)({}_pF_q(m) - 1)\right]\\
&\leq 1-\alpha,
\end{aligned}
$$

by the given hypothesis.

Thus, the proof of Theorem 5.3 is established. □

Theorem 5.4. If $a_i,\ b_j > 1$ $(i = 1,2,...,p;\ j = 1,2,...,q)$ then ${}_p\bar{I}_q(m,z)$ is in the class $T(\lambda,\alpha)$, if and only if the condition

$$
\begin{aligned}
&\alpha(1-\lambda) \frac{(b_1-1)...(b_q-1)}{(a_1-1)...(a_p-1)m} (1 - {}_pF_q(a_1-1,\ b_1-1;\ m))\\
&\leq 1 - \alpha - \alpha(1-\lambda)\ {}_pF_q(m)
\end{aligned}
$$

holds with either one of the conditions

1. $p \leq q$ and $m > 0$.
2. $p = q+1$ and $m < 1$.
3. $p = q+1,\ m = 1$ and $\sum_{j=1}^{q} b_j > \sum_{i=1}^{q} a_i$.

Proof. The proof of above theorem is much similar to that of Theorem 5.3. Therefore we omit the details involved. □

References

[1] M. S. Ahmad, Q. Mehmood, W. Nazeer and A. U. Haq, An application of hypergeometric distribution series on certain analytic functions, Sci. Int., 27 (2015), 2989-2992.

[2] O. Altintas and S. Owa, On subclasses of univalent functions with negative coefficients, Pusan Kyongam Math J., 4 (1988), 41-56.

[3] S. Altnkaya and S. Yalcin, Poisson distribution series for analytic univalent functions, Complex Analysis and Operator Theory, 12(5) (2018), 1315-1319.

[4] D. Bajpai, A study of univalent functions associated with distribution series and q- calculus, M.Phil. Dissertation, CSJM University, Kanpur, 2016.

[5] A. Baricz, Generalized Bessel functions of the first kind, Ph.D. Thesis, Babes-Bolyai Univ- Cluj- Napra 2009.

[6] T. Bulboaca and G. Murugusundramoorthy, Univalent functions with positive coefficients involving Pascal distribution series, Commun. Korean Math. Soc., 35(3) (2020), 867-877.

[7] K. K. Dixit and S. K. Pal, On a class of univalent functions related to complex order, Indian J. Pure Appl. Math., 26 (9) (1995), 889-896.

[8] W. Nazeer, Q. Mehmood, S. M. Kang and A. U. Haq, An application of Binomial distribution series on certain analytic functions, J. Comput. Anal. Appl, 26 (2019), 11-17.

[9] S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, Canad. J. Math., 39 (1987), 1057-1077.

[10] S. Ponnusamy and F. Rønning, Starlikeness properties for convolutions involving hypergeometric series, Ann. Univ. Mariae Curie-Sklodawask Sect. A., 52 (1998), 141-155.

[11] S. Ponnusamy and M. Vuarinen, Univalence and convexity properties for Gaussian hypergeometric functions, The Rocky Mountain J. Math., 31 (1) (2001), 327-353.

[12] S. Porwal, An application of Poisson distribution series on certain analytic functions, J. Complex Anal., Vol. 2014, Art ID 984135, 1-3.

[13] S. Porwal, Generalized distribution and its geometric properties associated with univalent functions, J. Complex Anal., Vol. 2018, Art ID 8654506, 1-5.

[14] Saurabh Porwal and D. Breaz, Mapping properties of an integral operator involving Bessel functions, Analytic Number Theory, Approximation Theory and Special Functions, Springer, New York, (2014), 821-826.

[15] S. Porwal and S. Kumar, Confluent hypergeometric distribution and its application on certain functions, classes of univalent functions, Afr. Mat., **28** (2017), no. 1-2, 1–8.

[16] S. Porwal and A. Gupta, Some properties of convolution for hypergeometric distribution type series on certain analytic univalent functions, Acta Univ. Apul., 56 (2018), 69-80.

[17] E. D. Rainville, Special Functions, The Macmillan Co., New York, 1960.

[18] R. Themangani, S. Porwal and N. Mangesh, Generalized hypergeometric distribution and its applications on univalent functions, J. Inequal. Appl., Accepted, (2020).

[19] A. K. Wanas and J. A. Khuttar, Applications of Borel distribution series on analytic functions, Earthline J. Math. Sci., 4 (1) (2020), 71-82.