Generalized $\alpha - \varphi$ contraction mapping theorem in cone metric spaces and application

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Abstract: In this paper firstly, define generalized $\alpha - \varphi$-contraction mappings in cone metric spaces which is generalization of metric spaces. Secondly, we prove fixed point theorems for generalized $\alpha - \varphi$-contraction mappings in cone metric spaces. Our results generalize and extend some well known results of [29].Some examples are presented to verify the effectiveness and applicability of our main results. Towards the end, an application to integral equations has also been presented to support the usability of the obtained results.

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I. Introduction

The first fundamental result on fixed point for contractive mapping was published and introduced by S. Banach [1] in 1922, which is known as Banach contraction mapping principle. It is widely recognized as influential sources in pure and applied Mathematics. Amapping$T: X \to X$, where(X, d) is a metric space, is said to be a contraction mapping, such that

$d(Tx, Ty) \leq kd(x, y)$, for all $x, y \in X$ with contractive constant $k \in [0,1)$.

This principle provides an impressive illustration of the unifying power of functional analytic methods and their usefulness in various disciplinary. It has become a very popular source of existence and uniqueness theorems in different branches of Mathematical analysis.

These sources have been associated with new and generalized classes of contractive mappings. In this direction, Samet et al. [2] introduced the concept of α −admissible, α −contractive, $\alpha - \varphi -$ contractive mapping and proved fixed point results, further he also extend to the(

$\alpha, \beta) -$contractive mappings in metric spaces. Such that, several authors obtained further results (see for instance [3-17]).

In 2007,Huang and Zhang [18] introduced cone metric space , which is generalization of metric space by replacing the real numbers with ordered Banach space and obtained fixed point theorems of contractive mappings in these spaces. Subsequently,several authors have been generalized and studied fixed and common fixed point results in cone metric spaces for normal and non normal (see for instance [19-28]).

Quite recently, Kang, S. M. et al. [29] introduced $\alpha - \varphi$-contractive mappings in cone metric spaces and proved fixed point theorem, which is generalization of theresult [2]. Further, Verma, M. et al. [30] generalized and extend the results [29] and proved fixed point theorem.

Thus, the purpose of this paper is to give the generalized version of $\alpha - \varphi$-Ccontractive type contraction mappings in cone metric spaces and get the fixed point theorem. Our results generalize and extend previously obtained result of [29].

2. Preliminaries

First, we introduce some standard notations and definitions which we needed them in the sequel see **([18]).**

Definition 2.1: Let E be a real Banach space and$P$ be a subset of $E$and 0denote to the zero element in$E$, then P is called a cone if and only if :

(i) P is a non-empty set closed and $P \neq \{0\}$,

(ii) If a, b are non-negative real numbers and $x, y \in P$,then $ax + by \in P$,

(iii) $x \in P and - x \in P \Longrightarrow x = 0 \Leftrightarrow P \cap (-P) = \{0\}$.

Given a cone P ⊂ E, we define a partial ordering ≤ on $E$with respect to $P$by$x \leq y$ if and only if $y - x \in P$.We shall write $x \ll y$ if $y - x \in intP$(where int P denotes the interior of P).If $intP \neq \emptyset$, then cone$P$ is solid. The cone P called normal if there is a number $M > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \Rightarrow \| x \| \leq M \| y \|.$$

Where M is least positive number satisfying the above is called the normal constant of P.

Definition2.2: Let $P \subset E$ is called regular if every increasing sequence which is bounded above is convergent .That is, if $\{x_n\}$ is sequence such that $x_1 \leq x_2 \leq x_3 \ldots\ldots\ldots \leq y$ for some $y \in E$,then there exist $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently the cone P is regular if and only if decreasing sequence which is bounded below is convergent.

Now for the following discussion assume that E is Banach space, P is a cone in E with $intP \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

Definition 2.3: Let X be a non-empty set. Suppose E is a real Banach space, P is a cone with $intP \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.If the mapping $d: X \times X \longrightarrow E$ satisfies

(i) $0 < d(x,y) forall x, y \in X and (x,y) = 0$if and only if$x = y$,

(ii) $d(x,y) = d(y,x) forall x, y \in X$,

(iii) $d(x,y) \leq d(x,z) + d(z,y) forall x, y, z \in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space. The concept of cone metric space is more general than that of a metric space.

Example 2.4: Let $E = R^2, P = \{(x,y) \in E: x, y \geq 0\}, X = R$ and $d: X \times X \to E$ defined by $d(x,y) = (|x-y|, \alpha|x-y|)$,where $\alpha \geq 0$ is a constant. Then (X,d) is a cone metric space.

Definition: 2.5: Let (X,d) be a cone metric space, $x \in X$ and $\{x_n\}_{n\geq 1}$ be a sequence in$X$. Then,

(1) $\{x_n\}_{n\geq 1}$Converges to x whenever for every$c \in E$ with $\theta \ll c$, if there is a natural Number Nsuch that$d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $lim_{n\to\infty} x_n = x or x_n \to x, (n \to \infty)$

(2) $\{x_n\}_{n\geq 1}$is said to be a Cauchy sequence if for every $c \in E$ with θ<<c, if there is Natural number$N$ such that$d(x_n, x_m) \ll c$ for all $n.m \geq N$.

(3) (X,d) is called a complete cone metric space if every Cauchy sequence in X is Convergent.

Lemma 2.6: Let (X,d) be a cone metric space and Pbe a normal cone with normal constant M. Let $\{x_n\}_{n\geq 1}$ be a sequence in$X$. Then,

(a) $\{x_n\}_{n\geq 1}$Converges to x if and only if $d(x_n, x) \to 0 (n \to \infty)$.

(b) $\{x_n\}_{n\geq 1}$is a Cauchy sequence if and only if $d(x_n, x_m) \to \infty (n, m \to \infty)$.

Lemma 2.7: Let (X, d) be a cone metric space and P be a normal cone with normal constant M. Let $\{x_n\}_{n\geq 1}$ be a sequence in$X$. Then,

(i) The limit $\{x_n\}_{n\geq 1}$ is unique. That is if $\{x_n\}_{n\geq 1}$Converges to x and$\{x_n\}_{n\geq 1}$Converges to y.

(ii) if $\{x_n\}_{n\geq 1}$Converges to x then $\{x_n\}_{n\geq 1}$is a Cauchy sequence.

Lemma 2.8: Let (X, d) be a cone metric space and P be a normal cone with normal constant M. Let $\{x_n\}_{n\geq 1}$ and$\{y_n\}_{n\geq 1}$be any two sequences in$X$with $\{x_n\}_{n\geq 1} \to x$ and $\{y_n\}_{n\geq 1} \to y$ as $n \to \infty$, then $d(x_n, y_n) = d(x, y)$.

Definition: 2.9[2]:Let $T: X \to X$ and $\alpha: X \times X \to [0, \infty)$, we say that T is α-adimissible if $\alpha d(x, y) \geq 1 => \alpha d(Tx, Ty) \geq 1$ for all $x, y \in X$.

We denote with Ψ the family of non-decreasing function $\varphi: [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \varphi^n < +\infty$ for each $t > 0$, where φ^n is n^{th}iterationφ.

Lemma 2.10[2]: For every $\varphi: [0, +\infty) \to [0, +\infty)$ the following holds:

If φ is non decreasing, then for each $t > 0, \lim_{n\to\infty} \varphi^n (t) = 0$ implies $\varphi(t) < t$ and $\varphi(0) = 0$.

Definition: 2.11[2]: Let $T: X \to X$ be a mapping in Metric space(X, d) is said to be an $\alpha - \varphi -$contractive mapping, if there exists two functions $\alpha: X \times X \to [0, \infty)$ and $\varphi \in \Psi$ such that $\alpha d(x, y) d(Tx, Ty) \leq \varphi d(x, y)$, for all $x, y \in X$.

Further **Kang et al.[29**] introduce the notion of this mapping in cone metric spaces as follows:

Definition 2.12: Let (X, d) be a cone metric space and P be a normal cone with normal constant M. Let $T: X \to X$ be a mapping. T is called $\alpha - \varphi -$contractive mapping, if there exists two functions $\alpha: X \times X \to [0, \infty)$and $\varphi \in \Psi$ suchthat

$\alpha d(x, y) d(Tx, Ty) \leq \varphi d(x, y)$, for all $x, y \in X$.

Now we present new notion of generalized $\alpha - \varphi -$contraction mappings in cone metric spaces and derive fixed point results for these mappings in cone metric space.

Definition 2.12: Let (X, d) be a cone metric space and P be a normal cone with normal constant M. Let $T: X \to X$ be a mapping. T is said to be generalized $\alpha - \varphi -$contraction mapping, if

There exist two functions $\alpha: X \times X \to [0, \infty)$ and $\varphi \in \Psi$ such that

$$\alpha d(x,y) d(Tx,Ty) \leq \varphi[\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)$$

for all $x, y \in X$.

3.Main Result

In this section we will extend and generalize of the given result of Kang et al. [29].

Kang et al. [29] proved the following theorem in cone metric spaces:

Theorem 3.1: Let (X, d) be a cone metric space and P be a normal cone with normal constant M. Let $T: X \to X$ be an $\alpha - \varphi$ −contractive mappingsatisfying the following conditions:

(i) T is α-adimissible ;

(ii) There exists$x_0 \in X$such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is continuous or

If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$for all $n \in N$, then T has a fixed point.

Now we prove theorem 3.1 in the setting of cone metric spaces for generalized $\alpha - \varphi$contractive mapping as follows:

Theorem 3.2: Let (X, d) be a cone metric space and P be a normal cone with normal constant M. Let $T: X \to X$ be generalized $\alpha - \varphi$ −contraction mappingby definition [2.12] satisfying the following conditions:

(iv) T is α-adimissible ;

(v) There exists$x_0 \in X$such that $\alpha(x_0, Tx_0) \geq 1$;

(vi) T is continuous or

If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in N$, then T has a fixed point.

Proof: Let $x_0 \in X$ such that that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ in X such that

$x_{n+1} = Tx_n = T^n x_0$, for some $n \in N$..................(3.1)

If $x_n = x_{n+1}$for some $n \in N$, then x_n is a fixed point of T. Assume that $x_n \neq x_{n+1}$ for all $n \in N$. Since T is α-adimissible, we have

$$\alpha(x_0, x_1) = \alpha(x_0, x_1) \geq 1 => \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

By induction, we get

$\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$.......................... (3.2)

Now applying inequality (2.12) and (3.2), we obtain

$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$

$\leq \alpha(x_{n-1}, x_n) d(Tx_{n-1}, Tx_n)$

$\leq \varphi[\max\{\mathrm{d}(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\}]$

$\leq \varphi[\max\{\mathrm{d}(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n)\}]$

$\leq \varphi[\max\{\mathrm{d}(x_{n-1}, x_n), d(x_n, x_{n+1})\}]$......................... (3.3)

If for some $n \geq 1$, we have $\mathrm{d}(x_{n-1}, x_n) < d(x_n, x_{n+1})$ .Then (3.3) becomes

$$d(x_n, x_{n+1}) \leq \varphi d(x_n, x_{n+1})$$

Which implies

$$\|d(x_n, x_{n+1})\| \leq \|\varphi d(x_n, x_{n+1})\|$$

$< \|d(x_n, x_{n+1}\|$. This is contradiction. Thus for all $n \geq 1$, we have

Max { $d(x_{n-1}, x_n), d(x_n, x_{n+1}) = d(x_{n-1}, x_n)$............................... (3.4)

In view of (3.3) and (3.4) we get

$d(x_n, x_{n+1}) \leq \varphi\, d(x_{n-1}, x_n)$............................... (3.5)

Continuing this process inductively, we obtain

$d(x_n, x_{n+1}) \leq \varphi^n (x_0, x_1)$ for all $n \in N$ .... (3.6)

So, for $m, n \in N$ with$n > m$ we have

$d(x_{n,} x_m) \leq d(x_{n,} x_{n-1}) + d(x_{n-1,} x_{n-2}) + \ldots\ldots + d(x_{m+1,} x_m)$

$\leq (\varphi^{n-1} + \varphi^{n-2} + \cdots \ldots \ldots \ldots \ldots \ldots + \varphi^m) d(x_{0,} x_1),$

$\leq \frac{\varphi^n}{1-\varphi} d(x_{0,} x_1)$ .... (3.7)

Since P is normal cone with normal constant, so by (3.7) we get

$\|d(x_{n,} x_m)\| \leq M[\frac{\varphi^n}{1-\varphi} \|d(x_{0,} x_1)\|] \to 0,$

which implies that $d(x_{0,} x_1) \to 0$ as $n, m \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence in the cone metric space(X, d). Since (X, d) is complete. So there exist $u \in X$ such that

$x_n \to u$ as $n \to \infty$.

Case-I. If T is continuous, then we have $x_{n+1} = Tx_n \to Tu$ as $n \to \infty$.By uniqueness of limit.$Tu = u$. Hence u is a fixed point of T.

Case–II. If $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \geq 1$,for all $n \in N$ and $x_n \to u$ as $n \to \infty$, then $\alpha(x_n, u) \geq 1$, for all $n \in N$.

Now, we show that $\|d(Tu, u)\| \geq 0$ as $n \to \infty$. On contrary, assume that $\|d(Tu, u\| \geq 0$.We have

$$d(Tu, u) \leq d(Tu, Tx_n) + d(x_{n+1,}u)$$

$$\leq \alpha(x_n, u)d(Tu, Tx_n) + d(x_{n+1,}u)$$

$$\leq \varphi\left[\max\left\{d(x_n, u), d(x_n, x_{n+1})d(u, Tu), d\left(x_{n,}Tu\right), d(u, x_{n+1})\right\}\right] + d(x_{n+1}, u)$$

$\leq \varphi(M) + d(x_{n+1,}\, u)$……………………………………………(3.8)

Where M =max$\left\{d(x_n, u), d(x_n, x_{n+1})d(u, Tu), d\left(x_{n,}Tu\right), d(u, x_{n+1})\right\}$

Since P is normal cone with normal constant M, we have

$\|d(Tu, u)\| \leq M[\|\varphi(M)\|] + \left\|d(x_{n+1,}\, u)\right\| \to 0$,……………………………(3.9)

Where M =max$\left\{d(x_n, u), d(x_n, x_{n+1})d(u, Tu), d\left(x_{n,}Tu\right), d(u, x_{n+1})\right\}$, Letting$n \to \infty$, we have

$$M = \varphi[d(u, Tu) + d(d(x_{n+1,}\, u)$$

Using (3.9) and taking $n \to \infty$, we have $\|d(Tu, u)\| \to 0$ implies that $d(Tu, u) = 0$. Thus $Tu = u$. Hence u is fixed point of T. This completes the proof.

Example3.3: Let us consider $X = [0, \infty)$ and $E = R^2, P = \{(x, y) \in E | x, y > 0\} \subset R^2$ and $d: X \times X \to E$ such that $d(x, y) = (|x - y|, a|x - y|)$ where $a \geq 0$ is a constant. Then (X, d)be a cone metric space. Define the mapping $T: X \to X$ by

$2x - \frac{7}{4}x > 1,$

$Tx = \frac{x}{4}$ if $0 \leq x \leq 1;$

if $x < 0.$

We observe that here T is continuous and Banach contraction principle in the setting of cone metric space cannotapply.

Now we a mapping $\alpha: X \times X \to [0, \infty)$ by

$e^{|x-y|}$, if $x, y \in [0, \frac{1}{4}]$

$\alpha(x, y) =$

$e^{-|x-y|}$,Otherwise

Clearly, T is generalized α-φcontraction mapping with $\varphi(t) = \frac{t}{4}$ for all $t \in [0, \infty)$. Infect for all $x, y \in X$, we have

$\alpha(x, y)d(Tx, Ty) \le \varphi[\max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}]$

More over there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. For $x_0 = \frac{1}{4}$, we have

$$\alpha\left(\frac{1}{4}, T_1\right) = \alpha\left(\frac{1}{4}, \frac{1}{16}\right)$$

$$= e^{3/16} \geq 1.$$

Now it remains to show that T is α-admossible. Let $x, y \in X$such that $\alpha(x, y) \geq 1$.therefore, we have$x, y \in [0, \frac{1}{4}]$. By definition T and α we have

$Tx = \frac{x}{4} \in \left[0, \frac{1}{4}\right], Ty = \frac{y}{4} \in [0, \frac{1}{4}]$ and $\alpha(Tx, Ty) = 1$.So, T is α-admissible.

Finally let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$ and $x_n \to \infty$. since $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$,by definition of α, we have $x_n \in [0,1]$. Then $\alpha(x_{n,}x) = 1$.

Now all hypothesis of theorem 3.2 are satisfied. Consequently, T has a fixed point. Recall that, theorem 3.2 guarantees only the existence of fixed point but not uniqueness. In this example 0 and $\frac{7}{4}$ are two fixed points of T.

To assure that the uniqueness of fixed point, we will consider the following hypothesis.

(*) for all $x, y \in X$thewre exists $z \in X$such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

Theorem 3.4: Theorem 3.2 yields a unique fixed point after adding hypothesis (*) to it.

Proof. Let u and v are two fixed point of T. From (*), there exist $z \in X$ such that

$\alpha(u, z) \geq 1$ and $\alpha(v, z) \geq 1$ ....……………………(3.10)

Define a sequence $\{z_n\}$ in X by $z_{n+1} = Tz_n$ for all$n \geq 0$ and $z_0 = z$.since$T$ is α-admissible, from the above inequalities, for all $n \in N$

$\alpha(u, z_n) \geq 1$ and $\alpha(v, z_n) \geq 1$ ....…………………(3.11)

Using inequalities (2.1) and (3.11), we get

$$d(u, z_{n+1}) \leq d(Tu, Tz_n)$$

$\leq \alpha(u, z_n)d(Tu, Tz_n)$

$$\leq \varphi[\max\{d(u, z_n), d(u, Tu), d(u, Tz_n), d(z_n, Tu)\}]$$

$\leq \varphi(N)$ …………………………………………………(3.12)

Where N = max$\{d(u,z_n), d(u,Tu), d(u,Tz_n), d(z_n,Tu)\}$

$\leq$max $\{d(u,z_n), d(u,z_{n+1})$ ……………………………….(3.13)

Now owing to the monotonicity of φ and using inequality (3.12), we have

$d(u,z_{n+1}) \leq \varphi$max $\{d(u,z_n), d(u,z_{n+1})$ for all $n \in N$, ……(3.14)

Without loss of generality, let$d(u,z_n) \geq 0$ for all $n \in N$. If

max$\{$d$(u,z_n), d(u,z_{n+1}) = d(u,z_{n+1})$. Then

$d(u,z_{n+1}) \leq \varphi\, d(u,z_{n+1})$.

$\|d(u,z_{n+1})\| \leq \|\varphi\, d(u,z_{n+1})\| < \|d(u,z_{n+1})\|$.

Which is contradiction. thus we have

max $\{$d$(u,z_n), d(u,z_{n+1}) = d(u,z_n)$………………………(3.15)

In view of (3.14) and (3.15) ,we get

$d(u,z_{n+1}) \leq d(u,z_n)$, for all $n \geq 1$.

Continuing this process inductively, we get

$d(u,z_n) = \varphi^n d(u,z_0)$, for all $n \geq 1$.

Since P be a normal cone with normal constant$M$, we have

$\|d(u,z_n)\| \leq M\|\varphi^n d(u,z_0\| \to 0$ as $n \to \infty$, which implies that$z_n \to u$ as $n \to \infty$,….(3.16)

Similarly, we can get $z_n \to v$ as $n \to \infty$,………………………… (3.17)

Using triangular inequality, we have

$\|d(u,v)\| \to 0$ as $n \to \infty$. Thus $d(u,v) = 0$, which implies that $u = v$. Hence T has the unique fixed point .This completes the proof.

4. Application

In this section we shall apply theorem 3.2 to the non linear quadratic integral equation

$x(t) = f(t) + \theta \int_0^t k(t,u) l(u,x(u)) du, t \in [0,T],\ T > 0$ ……. (3.18)

Let $X = C[0,T]$ be the set continuous function in [0,T] and

$$d(x,y) = \sup_{t\in[0,T]} |x(t) - y(t)|, x, y \in [0,T]$$

It is easy to see that (X,d) is complete metric space. We consider (3.18) under the following assumptions:

1. $f: [0,T] \to R$ is continuous;
2. $l: [0.T] \to R$ is continuous and for all $t \in [0,T], x \leq y$, we have $l(t,x) \leq l(t,y), |l(t,x) - l(t,y)| \leq N|x - y|$, where $N > 0$is a constant;

3. $k:[0,T]\times[0,T]\to[0,\infty)$ ios continuous and there exists a constant $M>0$ such that

$$\int_0^t k(t,u)|x(u)-y(u)|\ du \le M, t\in[0,T];$$

4. There exist $x_0 \in X$such that

$x_0(t) = f(t) + \theta \int_0^t k(t,u)l(u,x(u))du, t\in[0,T],\ T>0$;

We have the following theorem:

Theorem 3.5: Suppose the above condition (1) - (4) are satisfied, if $\theta KNT < 1$. Then the integral equation (3.18) has a unique continuous solution $x^* \in [0,T]$.

Proof: We consider the mapping $T: X \to X$ defined by

$Tx(t) = f(t) + \theta \int_0^t k(t,u)l(u,x(u))du, t\in[0,T],\ T>0$ ……………..(3.19)

Now, we show that T is α-φ generalized contractive mapping in cone metric spaces,i.e.

$\alpha(x,y)d(Tx,Ty) \le \varphi[U(x,y)]$……………………………… (3.20)

Where $U(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$

Now we let the function $\alpha: X\times X \to R$ defined by

$$\alpha(x,y)=\begin{cases} 1, & \text{if } x(t)\le y(t) \\ 0, & \text{otherwise} \end{cases} \quad t\in[0,T]$$

, and the function $\varphi(t):[0,\infty)\to[0,\infty)$ defined by $\varphi(t) = (\theta\ KNT)t, t\in[0,T]$. Obviously, $\varphi \in \Psi$.

$d(Tx(t),Ty(t)) = \text{Sup}_{t\in[0,T]}|x(t)-y(t)|, x,y \in [0,T]$…………..(3.21)

Also if $x(t)\le y(t)$ is not satisfied .then the inequality (3.20) holds immediately. So, we may Suppose $x(t)\le y(t), t\in[0,T]$. From (ii), (iii) and (3.19), we have

$|Tx(t)-Ty(t)| = \left|f(t) + \theta\int_0^t k(t,u)l\big(u,x(u)\big)du - f(t) - \theta\int_0^t k(t,u)l(u,y(u))du\right|$

$$\le \theta \int_0^t k(t,u)|l(u,x(u)) - (u,y(u))|du$$

$$\le \theta \int_0^t k(t,u)N|x(u)) - y(u))|du$$

$\le \theta KNT|x(u)) - y(u))|du.$

So, from (3.21), we get

$|Tx(t)-Ty(t)| \le \theta KNT d(x,y)$

$\le (\theta KNT)U(x,y)$ ……………………………(3.22)

Put $\theta KNT = 1$, and by (3.22) we get

$$\alpha(x,y)d(Tx(t) - Ty(t)) \leq \varphi[U(x,y)]$$

So, T is an α-φ generalized contractive mapping in cone metric spaces.

Put $x_n = T^n x_{0,}\ n \in N$ from condition (iv) we get

$\alpha(x_0, Tx_0) = 1$. and condition (ii) we get $\alpha(x,y) = 1 => \alpha(Tx,Ty) = 1$.

So, by induction, we get $\alpha(x_n, x_{n+1}) = 1$. Also from the proof of theorem 3.2, we know that$x_n \to u \in X$.Then $\alpha(x_n, u) = 1$. Hence all assumption of theorem 3.2 are satisfied. So, according to theorem 3.2 we can deduce that u is a fixed point of$T, i.e.\ u$ is a solution to the integral equation (3.19). Also take $u(t) = max\,\{x(t), y(t)\}, t \in [0,T]$. Then for all $x, y \in X$, there exist $\alpha(x,u) = \alpha(y,u) = 1$, from theorem (3.4), we know that u is the unique solution to the integral equation (3.19).

5. Conclusion

In this attempt, we establish new generalizeα-φcontraction mapping in complete cone metric spaces and fixed point. These results generalize, improve and extend the theorem [29]. Also give the example for verified the results with an application to integral equation is given here to illustrate the usability of the getting results.

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