

A STUDY ON BIPOLAR VALUED VAGUE SUBRINGS OF A RING

¹B. Deeba, ²S. Naganathan & ³K. Arjunan

¹Department of Mathematics, Idhaya College for Women, Sarugani – 630411, Tamilnadu, India.
Email: bdeeba85@gmail.com

²Department of Mathematics, Sethupathy Government Arts College, Ramanathapuram -623 502,
Tamilnadu, India. Email: nathanaga@gmail.com

³Department of Mathematics, Alagappa Government Arts college, Karaikudi – 630003, Tamilnadu, India.
Email: arjunan.karmegam@gmail.com

ABSTRACT: In this paper, bipolar valued vague subring of a ring is introduced and some properties are discussed. Bipolar valued vague subring of a ring is a generalized form of vague subring of the ring, vague subring of the ring is a generalized form of fuzzy subring of the ring and fuzzy subring of the ring is a generalized form of ring.

KEY WORDS: Fuzzy subset, vague subset, bipolar valued fuzzy subset, bipolar valued vague subset, bipolar valued vague subring, intersection, product, strongest, height.

INTRODUCTION: In 1965, Zadeh [13] introduced the notion of a fuzzy subset of a set, fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. Since then it has become a vigorous area of research in different domains, there have been a number of generalizations of this fundamental concept such as intuitionistic fuzzy sets, interval valued fuzzy sets, vague sets, soft sets etc. Grattan-Guinness [6] discussed about fuzzy membership mapped onto interval and many valued quantities. Vague set is an extension of fuzzy set and it is appeared as a unique case of context dependent fuzzy sets. The vague set was introduced by W.L.Gau and D.J.Buehrer [5]. Lee [7] introduced the notion of bipolar valued fuzzy sets. Bipolar valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. In a bipolar valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree $(0, 1]$ indicates that elements somewhat satisfy the property and the membership degree $[-1, 0)$ indicates that elements somewhat satisfy the implicit counter property. Bipolar valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different each other [7, 8]. Fuzzy subgroup was introduced by Azriel Rosenfeld [2]. RanjitBiswas [10] introduced the vague groups. Cicily Flora. S and Arockiarani.I [4] have introduced a new class of generalized bipolar vague sets. Anitha.M.S., et.al.[1] defined as bipolar valued fuzzy subgroups of a group and Balasubramanian.A et.al[3] have defined the bipolar interval valued fuzzy subgroups of a group. K.Murugalingam and K.Arjunan[9] have discussed about interval valued fuzzy subsemiring of a semiring and Somasundra Moorthy.M.G.,[11] gave a idea about the fuzzy ring. Bipolar valued multi fuzzy subsemirings of a semiring have been introduced by Yasodara.B and KE.Sathappan[12]. Here, the concept of bipolar valued vague subring of a ring is introduced and established some results.

1.PRELIMINARIES:

Definition 1.1. [13] Let X be any nonempty set. A mapping $M : X \rightarrow [0, 1]$ is called a fuzzy subset of X .

Definition 1.2. [5] A vague set A in the universe of discourse U is a pair $[t_A, I-f_A]$, where $t_A : U \rightarrow [0, 1]$ and $f_A : U \rightarrow [0, 1]$ are mappings, they are called truth membership function

and false membership function respectively. Here $t_A(x)$ is a lower bound of the grade of membership of x derived from the evidence for x and $f_A(x)$ is a lower bound on the negation of x derived from the evidence against x and $t_A(x) + f_A(x) \leq 1$, for all $x \in U$.

Definition 1.3. [5] The interval $[t_A(x), 1-f_A(x)]$ is called the vague value of x in A and it is denoted by $V_A(x)$, i.e., $V_A(x) = [t_A(x), 1-f_A(x)]$.

Example 1.4. $A = \{ \langle a, [0.2, 0.9] \rangle, \langle b, [0.3, 0.8] \rangle, \langle c, [0.5, 0.9] \rangle \}$ is a vague subset of $X = \{a, b, c\}$.

Definition 1.5. [7] A bipolar valued fuzzy set (BVFS) A in X is defined as an object of the form $A = \{ \langle x, A^+(x), A^-(x) \rangle / x \in X \}$, where $A^+ : X \rightarrow [0, 1]$ and $A^- : X \rightarrow [-1, 0]$. The positive membership degree $A^+(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar valued fuzzy set A and the negative membership degree $A^-(x)$ denotes the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar valued fuzzy set A .

Example 1.6. $A = \{ \langle a, 0.8, -0.4 \rangle, \langle b, 0.6, -0.3 \rangle, \langle c, 0.2, -0.9 \rangle \}$ is a bipolar valued fuzzy subset of $X = \{a, b, c\}$.

Definition 1.7. [4] A bipolar valued vague subset A in X is defined as an object of the form $A = \{ \langle x, [t_A^+(x), 1-f_A^+(x)], [-1-f_A^-(x), t_A^-(x)] \rangle / x \in X \}$, where $t_A^+ : X \rightarrow [0, 1]$, $f_A^+ : X \rightarrow [0, 1]$, $t_A^- : X \rightarrow [-1, 0]$ and $f_A^- : X \rightarrow [-1, 0]$ are mapping such that $t_A(x) + f_A(x) \leq 1$ and $-1 \leq t_A^- + f_A^-$. The positive interval membership degree $[t_A^+(x), 1-f_A^+(x)]$ denotes the satisfaction region of an element x to the property corresponding to a bipolar valued vague subset A and the negative interval membership degree $[-1-f_A^-(x), t_A^-(x)]$ denotes the satisfaction region of an element x to some implicit counter-property corresponding to a bipolar valued vague subset A . Bipolar valued vague subset A is denoted as $A = \{ \langle x, V_A^+(x), V_A^-(x) \rangle / x \in X \}$, where $V_A^+(x) = [t_A^+(x), 1-f_A^+(x)]$ and $V_A^-(x) = [-1-f_A^-(x), t_A^-(x)]$.

Note that. $[0] = [0, 0]$, $[1] = [1, 1]$ and $[-1] = [-1, -1]$.

Example 1.8. $[A] = \{ \langle a, [0.2, 0.4], [-0.6, -0.2] \rangle, \langle b, [0.3, 0.4], [-0.5, -0.3] \rangle, \langle c, [0.4, 0.6], [-0.7, -0.2] \rangle \}$ is a bipolar valued vague subset of $X = \{a, b, c\}$.

Definition 1.9. [4] Let $A = \langle V_A^+, V_A^- \rangle$ and $B = \langle V_B^+, V_B^- \rangle$ be two bipolar valued vague subsets of a set X . We define the following relations and operations:

- (i) $[A] \subset [B]$ if and only if $V_A^+(u) \leq V_B^+(u)$ and $V_A^-(u) \geq V_B^-(u)$, $\forall u \in X$.
- (ii) $[A] = [B]$ if and only if $V_A^+(u) = V_B^+(u)$ and $V_A^-(u) = V_B^-(u)$, $\forall u \in X$.
- (iii) $[A] \cap [B] = \{ \langle u, \text{rmin}(V_A^+(u), V_B^+(u)), \text{rmax}(V_A^-(u), V_B^-(u)) \rangle / u \in X \}$.
- (iv) $[A] \cup [B] = \{ \langle u, \text{rmax}(V_A^+(u), V_B^+(u)), \text{rmin}(V_A^-(u), V_B^-(u)) \rangle / u \in X \}$. Here $\text{rmin}(V_A^+(u), V_B^+(u)) = [\min\{t_A^+(x), t_B^+(x)\}, \min\{1-f_A^+(x), 1-f_B^+(x)\}]$, $\text{rmax}(V_A^-(u), V_B^-(u)) = [\max\{t_A^-(x), t_B^-(x)\}, \max\{-1-f_A^-(x), -1-f_B^-(x)\}]$.

Definition 1.10. Let R be a ring. A bipolar valued vague subset A of R is said to be a bipolar valued vague subring of R (BVVSR) if the following conditions are satisfied,

- (i) $V_A^+(x-y) \geq \text{rmin}\{V_A^+(x), V_A^+(y)\}$
- (ii) $V_A^+(xy) \geq \text{rmin}\{V_A^+(x), V_A^+(y)\}$

(iii) $V_A^-(x-y) \leq \text{rmax}\{V_A^-(x), V_A^-(y)\}$

(iv) $V_A^-(xy) \leq \text{rmax}\{V_A^-(x), V_A^-(y)\}$ for all x and y in R .

Example 1.11. Let $R = Z_3 = \{0, 1, 2\}$ be a ring with respect to the ordinary addition and multiplication. Then $A = \{< 0, [0.6, 0.9], [-0.8, -0.5]>, < 1, [0.3, 0.6], [-0.7, -0.4]>, < 2, [0.3, 0.6], [-0.7, -0.4]>\}$ is a bipolar valued vague subring of R .

Definition 1.12. Let $A = \langle V_A^+, V_A^- \rangle$ and $B = \langle V_B^+, V_B^- \rangle$ be any two bipolar valued vague subsets of sets G and H , respectively. The product of A and B , denoted by $A \times B$, is defined as $A \times B = \{ \langle (x, y), V_{A \times B}^+(x, y), V_{A \times B}^-(x, y) \rangle / \text{for all } x \text{ in } G \text{ and } y \text{ in } H \}$ where $V_{A \times B}^+(x, y) = \text{rmin}\{V_A^+(x), V_B^+(y)\}$ and $V_{A \times B}^-(x, y) = \text{rmax}\{V_A^-(x), V_B^-(y)\}$ for all x in G and y in H .

Definition 1.13. Let $A = \langle V_A^+, V_A^- \rangle$ be a bipolar valued vague subset in a set S , the strongest bipolar valued vague relation on S , that is a bipolar valued vague relation on A is $V = \{ \langle (x, y), V_V^+(x, y), V_V^-(x, y) \rangle / x \text{ and } y \text{ in } S \}$ given by $V_V^+(x, y) = \text{rmin}\{V_A^+(x), V_A^+(y)\}$ and $V_V^-(x, y) = \text{rmax}\{V_A^-(x), V_A^-(y)\}$ for all x and y in S .

Definition 1.14. Let $A = \langle V_A^+, V_A^- \rangle$ be a bipolar valued vague subset of X . Then the height $H(A) = \langle H(V_A^+), H(V_A^-) \rangle$ is defined as $H(V_A^+) = \text{rsup}V_A^+(x)$ for all x in X and $H(V_A^-) = \text{rinf}V_A^-(x)$ for all x in X .

Definition 1.15. Let $A = \langle V_A^+, V_A^- \rangle$ be a bipolar valued vague subset of X . Then ${}^\oplus A = \langle {}^\oplus V_A^+, {}^\oplus V_A^- \rangle$ is defined as ${}^\oplus V_A^+(x) = V_A^+(x) + [1] - H(V_A^+)$ and ${}^\oplus V_A^-(x) = V_A^-(x) - [1] - H(V_A^-)$ for all x in X .

2. THEOREMS.

Theorem 2.1. Let $A = \langle V_A^+, V_A^- \rangle$ be a bipolar valued vague subring of a ring R . Then $V_A^+(-x) = V_A^+(x)$, $V_A^-(-x) = V_A^-(x)$, $V_A^+(x) \leq V_A^+(e)$, $V_A^-(x) \geq V_A^-(e)$, for all x in R , where e is the identity element in R .

Proof. Let x be in R . Now, $V_A^+(x) = V_A^+(-(-x)) \geq V_A^+(-x) \geq V_A^+(x)$ for all x in R . And $V_A^-(x) = V_A^-(-(-x)) \leq V_A^-(-x) \leq V_A^-(x)$ for all x in R . And $V_A^+(e) = V_A^+(x-x) \geq \text{rmin}\{V_A^+(x), V_A^+(-x)\} = V_A^+(x)$ for all x in R . And $V_A^-(e) = V_A^-(x-x) \leq \text{rmax}\{V_A^-(x), V_A^-(-x)\} = V_A^-(x)$ for all x in R .

Theorem 2.2. Let $A = \langle V_A^+, V_A^- \rangle$ be a bipolar valued vague subring of a ring R .

(i) If $V_A^+(x-y) = [0]$ then either $V_A^+(x) = [0]$ or $V_A^+(y) = [0]$ for x, y in R . (ii) If $V_A^+(xy) = [0]$ then either $V_A^+(x) = [0]$ or $V_A^+(y) = [0]$ for x, y in R . (iii) If $V_A^-(x-y) = [0]$ then either $V_A^-(x) = [0]$ or $V_A^-(y) = [0]$ for x, y in R . (iv) If $V_A^-(xy) = [0]$ then either $V_A^-(x) = [0]$ or $V_A^-(y) = [0]$ for x, y in R .

Proof. Let x, y in R . (i) By the definition $V_A^+(x-y) \geq \text{rmin}\{V_A^+(x), V_A^+(y)\}$ which implies that $[0] \geq \text{rmin}\{V_A^+(x), V_A^+(y)\}$. Therefore either $V_A^+(x) = [0]$ or $V_A^+(y) = [0]$. (ii) By the definition $V_A^+(xy) \geq \text{rmin}\{V_A^+(x), V_A^+(y)\}$ which implies that $[0] \geq \text{rmin}\{V_A^+(x), V_A^+(y)\}$. Therefore either $V_A^+(x) = [0]$ or $V_A^+(y) = [0]$. (iii) By the definition $V_A^-(x-y) \leq \text{rmax}\{V_A^-(x), V_A^-(y)\}$ which implies that $[0] \leq \text{rmax}\{V_A^-(x), V_A^-(y)\}$. Therefore either $V_A^-(x)$

$= [0]$ or $V_A^-(y) = [0]$. (iv) By the definition $V_A^-(xy) \leq \text{rmax} \{ V_A^-(x), V_A^-(y) \}$ which implies that $[0] \leq \text{rmax} \{ V_A^-(x), V_A^-(y) \}$. Therefore either $V_A^-(x) = [0]$ or $V_A^-(y) = [0]$.

Theorem 2.3. If $A = \langle V_A^+, V_A^- \rangle$ is a bipolar valued vague subring of a ring R , then $H = \{x \in R / V_A^+(x) = [1], V_A^-(x) = [-1]\}$ is either empty or a subring of R .

Proof. If no element satisfies this condition then H is empty. If x and y in H then $V_A^+(x-y) \geq \text{rmin} \{ V_A^+(x), V_A^+(y) \} = \text{rmin} \{ [1], [1] \} = [1]$. Therefore $V_A^+(x-y) = [1]$. And $V_A^+(xy) \geq \text{rmin} \{ V_A^+(x), V_A^+(y) \} = \text{rmin} \{ [1], [1] \} = [1]$. Therefore $V_A^+(xy) = [1]$. Also $V_A^-(x-y) \leq \text{rmax} \{ V_A^-(x), V_A^-(y) \} = \text{rmax} \{ [-1], [-1] \} = [-1]$. Therefore $V_A^-(x-y) = [-1]$. And $V_A^-(xy) \leq \text{rmax} \{ V_A^-(x), V_A^-(y) \} = \text{rmax} \{ [-1], [-1] \} = [-1]$. Therefore $V_A^-(xy) = [-1]$. That is $x-y \in H, xy \in H$. Hence H is a subring of R . Hence H is either empty or a subring of R .

Theorem 2.4. If $A = \langle V_A^+, V_A^- \rangle$ and $B = \langle V_B^+, V_B^- \rangle$ are two bipolar valued vague subrings of a ring R , then their intersection $A \cap B$ is a bipolar valued vague subring of R .

Proof. Let $C = A \cap B$ and let x, y in R . Now $V_C^+(x-y) = \text{rmin} \{ V_A^+(x-y), V_B^+(x-y) \} \geq \text{rmin} \{ \text{rmin} \{ V_A^+(x), V_A^+(y) \}, \text{rmin} \{ V_B^+(x), V_B^+(y) \} \} \geq \text{rmin} \{ \text{rmin} \{ V_A^+(x), V_B^+(x) \}, \text{rmin} \{ V_A^+(y), V_B^+(y) \} \} = \text{rmin} \{ V_C^+(x), V_C^+(y) \}$. Therefore $V_C^+(x-y) \geq \text{rmin} \{ V_C^+(x), V_C^+(y) \}$, for all x, y in R . And $V_C^+(xy) = \text{rmin} \{ V_A^+(xy), V_B^+(xy) \} \geq \text{rmin} \{ \text{rmin} \{ V_A^+(x), V_A^+(y) \}, \text{rmin} \{ V_B^+(x), V_B^+(y) \} \} \geq \text{rmin} \{ \text{rmin} \{ V_A^+(x), V_B^+(x) \}, \text{rmin} \{ V_A^+(y), V_B^+(y) \} \} = \text{rmin} \{ V_C^+(x), V_C^+(y) \}$. Therefore $V_C^+(xy) \geq \text{rmin} \{ V_C^+(x), V_C^+(y) \}$, for all x, y in R . Also $V_C^-(x-y) = \text{rmax} \{ V_A^-(x-y), V_B^-(x-y) \} \leq \text{rmax} \{ \text{rmax} \{ V_A^-(x), V_A^-(y) \}, \text{rmax} \{ V_B^-(x), V_B^-(y) \} \} \leq \text{rmax} \{ \text{rmax} \{ V_A^-(x), V_B^-(x) \}, \text{rmax} \{ V_A^-(y), V_B^-(y) \} \} = \text{rmax} \{ V_C^-(x), V_C^-(y) \}$. Therefore $V_C^-(x-y) \leq \text{rmax} \{ V_C^-(x), V_C^-(y) \}$, for all x, y in R . And $V_C^-(xy) = \text{rmax} \{ V_A^-(xy), V_B^-(xy) \} \leq \text{rmax} \{ \text{rmax} \{ V_A^-(x), V_A^-(y) \}, \text{rmax} \{ V_B^-(x), V_B^-(y) \} \} \leq \text{rmax} \{ \text{rmax} \{ V_A^-(x), V_B^-(x) \}, \text{rmax} \{ V_A^-(y), V_B^-(y) \} \} = \text{rmax} \{ V_C^-(x), V_C^-(y) \}$. Therefore $V_C^-(xy) \leq \text{rmax} \{ V_C^-(x), V_C^-(y) \}$, for all x, y in R . Hence $A \cap B$ is a bipolar valued vague subring of R .

Theorem 2.5. The intersection of a family of bipolar valued vague subrings of a ring R is a bipolar valued vague subring of R .

Proof. The proof follows from the Theorem 2.4.

Theorem 2.6. If $A = \langle V_A^+, V_A^- \rangle$ and $B = \langle V_B^+, V_B^- \rangle$ are any two bipolar valued vague subrings of the rings R_1 and R_2 respectively, then $A \times B = \langle V_{A \times B}^+, V_{A \times B}^- \rangle$ is a bipolar valued vague subring of $R_1 \times R_2$.

Proof. Let x_1, x_2 be in R_1, y_1 and y_2 be in R_2 . Then (x_1, y_1) and (x_2, y_2) are in $R_1 \times R_2$. Now, $V_{A \times B}^+ [(x_1, y_1) - (x_2, y_2)] = V_{A \times B}^+ (x_1 - x_2, y_1 - y_2) = \text{rmin} \{ V_A^+(x_1 - x_2), V_B^+(y_1 - y_2) \} \geq \text{rmin} \{ \text{rmin} \{ V_A^+(x_1), V_A^+(x_2) \}, \text{rmin} \{ V_B^+(y_1), V_B^+(y_2) \} \} = \text{rmin} \{ \text{rmin} \{ V_A^+(x_1), V_B^+(y_1) \}, \text{rmin} \{ V_A^+(x_2), V_B^+(y_2) \} \} = \text{rmin} \{ V_{A \times B}^+(x_1, y_1), V_{A \times B}^+(x_2, y_2) \}$. Therefore $V_{A \times B}^+ [(x_1, y_1) - (x_2, y_2)] \geq \text{rmin} \{ V_{A \times B}^+(x_1, y_1), V_{A \times B}^+(x_2, y_2) \}$. And $V_{A \times B}^+ [(x_1, y_1)(x_2, y_2)] = V_{A \times B}^+ (x_1 x_2, y_1 y_2) = \text{rmin} \{ V_A^+(x_1 x_2), V_B^+(y_1 y_2) \} \geq \text{rmin} \{ \text{rmin} \{ V_A^+(x_1), V_A^+(x_2) \}, \text{rmin} \{ V_B^+(y_1), V_B^+(y_2) \} \} = \text{rmin} \{ \text{rmin} \{ V_A^+(x_1), V_B^+(y_1) \}, \text{rmin} \{ V_A^+(x_2), V_B^+(y_2) \} \} = \text{rmin} \{ V_{A \times B}^+(x_1, y_1), V_{A \times B}^+(x_2, y_2) \}$.

Therefore $V_{A \times B}^+ [(x_1, y_1)(x_2, y_2)] \geq \text{rmin}\{V_{A \times B}^+(x_1, y_1), V_{A \times B}^+(x_2, y_2)\}$. Also $V_{A \times B}^- [(x_1, y_1) - (x_2, y_2)] = V_{A \times B}^- (x_1 - x_2, y_1 - y_2) = \text{rmax}\{V_A^-(x_1 - x_2), V_B^-(y_1 - y_2)\} \leq \text{rmax}\{\text{rmax}\{V_A^-(x_1), V_A^-(x_2)\}, \text{rmax}\{V_B^-(y_1), V_B^-(y_2)\}\} = \text{rmax}\{\text{rmax}\{V_A^-(x_1), V_B^-(y_1)\}, \text{rmax}\{V_A^-(x_2), V_B^-(y_2)\}\} = \text{rmax}\{V_{A \times B}^-(x_1, y_1), V_{A \times B}^-(x_2, y_2)\}$. Therefore $V_{A \times B}^- [(x_1, y_1) - (x_2, y_2)] \leq \text{rmax}\{V_{A \times B}^-(x_1, y_1), V_{A \times B}^-(x_2, y_2)\}$. And $V_{A \times B}^- [(x_1, y_1)(x_2, y_2)] = V_{A \times B}^- (x_1 x_2, y_1 y_2) = \text{rmax}\{V_A^-(x_1 x_2), V_B^-(y_1 y_2)\} \leq \text{rmax}\{\text{rmax}\{V_A^-(x_1), V_A^-(x_2)\}, \text{rmax}\{V_B^-(y_1), V_B^-(y_2)\}\} = \text{rmax}\{\text{rmax}\{V_A^-(x_1), V_B^-(y_1)\}, \text{rmax}\{V_A^-(x_2), V_B^-(y_2)\}\} = \text{rmax}\{V_{A \times B}^-(x_1, y_1), V_{A \times B}^-(x_2, y_2)\}$. Therefore $V_{A \times B}^- [(x_1, y_1)(x_2, y_2)] \leq \text{rmax}\{V_{A \times B}^-(x_1, y_1), V_{A \times B}^-(x_2, y_2)\}$. Hence $A \times B$ is a bipolar valued vague subring of $R_1 \times R_2$.

Theorem 2.7. Let $A = \langle V_A^+, V_A^- \rangle$ be a bipolar valued vague subset of a semiring R and $V = \langle V_V^+, V_V^- \rangle$ be the strongest bipolar valued vague relation of R . Then A is a bipolar valued vague subring of R if and only if V is a bipolar valued vague subring of $R \times R$.

Proof. Suppose that A is a bipolar valued vague subring of R . Then for any $x = (x_1, x_2), y = (y_1, y_2)$ are in $R \times R$. Now $V_V^+(x-y) = V_V^+ [(x_1, x_2) - (y_1, y_2)] = V_V^+ (x_1 - y_1, x_2 - y_2) = \text{rmin}\{V_A^+(x_1 - y_1), V_A^+(x_2 - y_2)\} \geq \text{rmin}\{\text{rmin}\{V_A^+(x_1), V_A^+(y_1)\}, \text{rmin}\{V_A^+(x_2), V_A^+(y_2)\}\} = \text{rmin}\{\text{rmin}\{V_A^+(x_1), V_A^+(x_2)\}, \text{rmin}\{V_A^+(y_1), V_A^+(y_2)\}\} = \text{rmin}\{V_V^+(x_1, x_2), V_V^+(y_1, y_2)\} = \text{rmin}\{V_V^+(x), V_V^+(y)\}$. Therefore $V_V^+(x-y) \geq \text{rmin}\{V_V^+(x), V_V^+(y)\}$ for all x, y in $R \times R$. And $V_V^+(xy) = V_V^+ [(x_1, x_2)(y_1, y_2)] = V_V^+ (x_1 y_1, x_2 y_2) = \text{rmin}\{V_A^+(x_1 y_1), V_A^+(x_2 y_2)\} \geq \text{rmin}\{\text{rmin}\{V_A^+(x_1), V_A^+(y_1)\}, \text{rmin}\{V_A^+(x_2), V_A^+(y_2)\}\} = \text{rmin}\{\text{rmin}\{V_A^+(x_1), V_A^+(x_2)\}, \text{rmin}\{V_A^+(y_1), V_A^+(y_2)\}\} = \text{rmin}\{V_V^+(x_1, x_2), V_V^+(y_1, y_2)\} = \text{rmin}\{V_V^+(x), V_V^+(y)\}$. Therefore $V_V^+(xy) \geq \text{rmin}\{V_V^+(x), V_V^+(y)\}$ for all x and y in $R \times R$. Also we have $V_V^-(x-y) = V_V^- [(x_1, x_2) - (y_1, y_2)] = V_V^- (x_1 - y_1, x_2 - y_2) = \text{rmax}\{V_A^-(x_1 - y_1), V_A^-(x_2 - y_2)\} \leq \text{rmax}\{\text{rmax}\{V_A^-(x_1), V_A^-(y_1)\}, \text{rmax}\{V_A^-(x_2), V_A^-(y_2)\}\} = \text{rmax}\{\text{rmax}\{V_A^-(x_1), V_A^-(x_2)\}, \text{rmax}\{V_A^-(y_1), V_A^-(y_2)\}\} = \text{rmax}\{V_V^-(x_1, x_2), V_V^-(y_1, y_2)\} = \text{rmax}\{V_V^-(x), V_V^-(y)\}$. Therefore $V_V^-(x-y) \leq \text{rmax}\{V_V^-(x), V_V^-(y)\}$ for all x, y in $R \times R$. And $V_V^-(xy) = V_V^- [(x_1, x_2)(y_1, y_2)] = V_V^- (x_1 y_1, x_2 y_2) = \text{rmax}\{V_A^-(x_1 y_1), V_A^-(x_2 y_2)\} \leq \text{rmax}\{\text{rmax}\{V_A^-(x_1), V_A^-(y_1)\}, \text{rmax}\{V_A^-(x_2), V_A^-(y_2)\}\} = \text{rmax}\{\text{rmax}\{V_A^-(x_1), V_A^-(x_2)\}, \text{rmax}\{V_A^-(y_1), V_A^-(y_2)\}\} = \text{rmax}\{V_V^-(x_1, x_2), V_V^-(y_1, y_2)\} = \text{rmax}\{V_V^-(x), V_V^-(y)\}$. Therefore $V_V^-(xy) \leq \text{rmax}\{V_V^-(x), V_V^-(y)\}$ for all x, y in $R \times R$. This proves that V is a bipolar valued vague subring of $R \times R$. Conversely assume that V is a bipolar valued vague subring of $R \times R$, then for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $R \times R$, we have $\text{rmin}\{V_A^+(x_1 - y_1), V_A^+(x_2 - y_2)\} = V_V^+(x_1 - y_1, x_2 - y_2) = V_V^+ [(x_1, x_2) - (y_1, y_2)] = V_V^+(x-y) \geq \text{rmin}\{V_V^+(x), V_V^+(y)\} = \text{rmin}\{V_V^+(x_1, x_2), V_V^+(y_1, y_2)\} = \text{rmin}\{\text{rmin}\{V_A^+(x_1), V_A^+(x_2)\}, \text{rmin}\{V_A^+(y_1), V_A^+(y_2)\}\}$. If $x_2 = y_2 = e$, we get, $V_A^+(x_1 - y_1) \geq \text{rmin}\{V_A^+(x_1), V_A^+(y_1)\}$ for all x_1 and y_1 in R . And $\text{rmin}\{V_A^+(x_1 y_1), V_A^+(x_2 y_2)\} = V_V^+(x_1 y_1, x_2 y_2) = V_V^+ [(x_1, x_2)(y_1, y_2)] = V_V^+(xy) \geq \text{rmin}\{V_V^+(x), V_V^+(y)\} = \text{rmin}\{V_V^+(x_1, x_2), V_V^+(y_1, y_2)\} = \text{rmin}\{\text{rmin}\{V_A^+(x_1), V_A^+(x_2)\}, \text{rmin}\{V_A^+(y_1), V_A^+(y_2)\}\}$. If $x_2 = y_2 = e$, we get $V_A^+(x_1 y_1) \geq \text{rmin}\{V_A^+(x_1), V_A^+(y_1)\}$ for all x_1 and y_1 in R .

Also we have $\text{rmax} \{ V_A^-(x_1-y_1), V_A^-(x_2-y_2) \} = V_V^-(x_1-y_1, x_2-y_2) = V_V^-[(x_1, x_2) - (y_1, y_2)] = V_V^-(x-y) \leq \text{rmax} \{ V_V^-(x), V_V^-(y) \} = \text{rmax} \{ V_V^-(x_1, x_2), V_V^-(y_1, y_2) \} = \text{rmax} \{ \text{rmax} \{ V_A^-(x_1), V_A^-(x_2) \}, \text{rmax} \{ V_A^-(y_1), V_A^-(y_2) \} \}$. If $x_2 = y_2 = e$, we get $V_A^-(x_1-y_1) \leq \text{rmax} \{ V_A^-(x_1), V_A^-(y_1) \}$ for all x_1 and y_1 in R . And $\text{rmax} \{ V_A^-(x_1y_1), V_A^-(x_2y_2) \} = V_V^-(x_1y_1, x_2y_2) = V_V^-[(x_1, x_2)(y_1, y_2)] = V_V^-(xy) \leq \text{rmax} \{ V_V^-(x), V_V^-(y) \} = \text{rmax} \{ V_V^-(x_1, x_2), V_V^-(y_1, y_2) \} = \text{rmax} \{ \text{rmax} \{ V_A^-(x_1), V_A^-(x_2) \}, \text{rmax} \{ V_A^-(y_1), V_A^-(y_2) \} \}$. If $x_2 = y_2 = e$, we get $V_A^-(x_1y_1) \leq \text{rmax} \{ V_A^-(x_1), V_A^-(y_1) \}$ for all x_1 and y_1 in R . Hence A is a bipolar valued vague subring of R .

Theorem 2.8. If $A = \langle V_A^+, V_A^- \rangle$ is a bipolar valued vague subring of a ring R , then ${}^\oplus A = \langle {}^\oplus V_A^+, {}^\oplus V_A^- \rangle$ is a bipolar valued vague subring of the ring R .

Proof. Let x and y in R . Now ${}^\oplus V_A^+(x-y) = V_A^+(x-y) + [1] - H(V_A^+) \geq \text{rmin} \{ V_A^+(x), V_A^+(y) \} + [1] - H(V_A^+) = \text{rmin} \{ V_A^+(x) + [1] - H(V_A^+), V_A^+(y) + [1] - H(V_A^+) \} = \text{rmin} \{ {}^\oplus V_A^+(x), {}^\oplus V_A^+(y) \}$ which implies ${}^\oplus V_A^+(x-y) \geq \text{rmin} \{ {}^\oplus V_A^+(x), {}^\oplus V_A^+(y) \}$ for all x and y in R . And ${}^\oplus V_A^+(xy) = V_A^+(xy) + [1] - H(V_A^+) \geq \text{rmin} \{ V_A^+(x), V_A^+(y) \} + [1] - H(V_A^+) = \text{rmin} \{ V_A^+(x) + [1] - H(V_A^+), V_A^+(y) + [1] - H(V_A^+) \} = \text{rmin} \{ {}^\oplus V_A^+(x), {}^\oplus V_A^+(y) \}$ which implies ${}^\oplus V_A^+(xy) \geq \text{rmin} \{ {}^\oplus V_A^+(x), {}^\oplus V_A^+(y) \}$ for all x, y in R . Also ${}^\oplus V_A^-(x-y) = V_A^-(x-y) - [1] - H(V_A^-) \leq \text{rmax} \{ V_A^-(x), V_A^-(y) \} - [1] - H(V_A^-) = \text{rmax} \{ V_A^-(x) - [1] - H(V_A^-), V_A^-(y) - [1] - H(V_A^-) \} = \text{rmax} \{ {}^\oplus V_A^-(x), {}^\oplus V_A^-(y) \}$ which implies ${}^\oplus V_A^-(x-y) \leq \text{rmax} \{ {}^\oplus V_A^-(x), {}^\oplus V_A^-(y) \}$ for all x, y in R . And ${}^\oplus V_A^-(xy) = V_A^-(xy) - [1] - H(V_A^-) \leq \text{rmax} \{ V_A^-(x), V_A^-(y) \} - [1] - H(V_A^-) = \text{rmax} \{ V_A^-(x) - [1] - H(V_A^-), V_A^-(y) - [1] - H(V_A^-) \} = \text{rmax} \{ {}^\oplus V_A^-(x), {}^\oplus V_A^-(y) \}$ which implies ${}^\oplus V_A^-(xy) \leq \text{rmax} \{ {}^\oplus V_A^-(x), {}^\oplus V_A^-(y) \}$ for all x, y in R . Hence ${}^\oplus A$ is a bipolar valued vague subring of R .

Theorem 2.9. Let $A = \langle V_A^+, V_A^- \rangle$ be a bipolar valued vague subring of a ring R . Then (i) $H(V_A^+) = [1]$ if and only if ${}^\oplus V_A^+(x) = V_A^+(x)$ for all x in R
(ii) $H(V_A^-) = [-1]$ if and only if ${}^\oplus V_A^-(x) = V_A^-(x)$ for all x in R .
(iii) ${}^\oplus V_A^+(x) = [1]$ if and only if $H(V_A^+) = V_A^+(x)$ for all x in R
(iv) ${}^\oplus V_A^-(x) = [-1]$ if and only if $H(V_A^-) = V_A^-(x)$ for all x in R .
(v) ${}^\oplus ({}^\oplus A) = {}^\oplus A$.

Proof. The proof is trivial.

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