DOUBT INTUITIONISTIC FUZZY H-IDEALS IN BH-ALGEBRAS

V.K.NIVEDIDA [1] and M.PALANIVELRAJAN [2]

[1] V.K.NIVEDIDA
Department of Mathematics
Noble College of Arts and Science for Women,
Palavanatham - 626 004.
e-mail: vknivedida1993@gmail.com

[2] M. PALANIVELRAJAN
Department of Mathematics,
Govt. Arts College,
Melur - 625 106.
e-mail: palanivelrajan1975@gmail.com

Abstract: In this paper we introduce the notion of a doubt intuitionistic fuzzy H-ideals in BH-algebras, and investigate some properties of it.
Keywords: BH-algebra, Intuitionistic fuzzy H-ideal, Doubt intuitionistic fuzzy H-ideal.

1. **Introduction**

The study of BH-algebra was initiated by Jun, Roh and Kim as a generalisation of BCH-algebras in 1998 [3]. After the introduction of fuzzy sets by Zadeh [4], there has been a number of generalisation of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [1,2] is one among them. To develop the theory of BH-algebras the ideal theory plays an important role. Several researchers investigated properties of fuzzy subalgebra and ideals in BCK/BCI-algebras [3,4]. In 2006, Park [4] introduced the concept of an interval-valued fuzzy ideal in BH-algebras. Following [7], we are going to introduce the concept of doubt intuitionistic fuzzy H-ideals in BH-algebras. After a detailed study of its properties, we come to this conclusion that in BH-algebras, an intuitionistic fuzzy subset is a doubt intuitionistic fuzzy H-ideal if and only if the complement of this intuitionistic fuzzy subset is an intuitionistic fuzzy H-ideal. Relations among doubt intuitionistic fuzzy ideals and doubt intuitionistic fuzzy H-ideals are also finally investigated.

2. Preliminaries

In this section, some elementary aspects that are necessary for this paper are included.

Definition 2.1. [3] Let X be a non-empty set with a constant 0 and a binary operation $*$.Then $(X, *, 0)$ is called a BH-algebra if it satisfies the following conditions

(1) $x * x = 0$,

(2) $x * y = 0$ and $y * x = 0 \Rightarrow x = y$,

(3) $x * 0 = x$ for all $x, y \in X$.

Definition 2.2. If a BH-algebra satisfies $(x * y) * z = x * (y * z)$ for all $x, y, z \in$ X, then it is called associative.
A partial ordering " $\leq$ " on a BH-algebra X can be defined by $x \leq y$ if and only if $x * y = 0$.
Throughout this paper, X always means a BH-algebra without any specification.

Definition 2.3. [4] A fuzzy set $H = \{\langle x, \delta_H(x)\rangle : x \in X\}$ of a BH-algebra in X is called a fuzzy ideal of X if it satisfies

(1) $\delta_H(0) \geq \delta_H(x)$ and

(2) $\delta_H(x) \geq min\{\delta_H(x * y), \delta_H(y)\}$ for all $x, y \in X$.

Throughout this paper, X always means a BH-algebra without any specification.

Definition 2.4. A fuzzy set δ in a BH-algebra X is called a fuzzy H-ideal of X if

(1) $\delta(0) \geq \delta(x)$ and

(2) $\delta(x * z) \geq min\{\delta(x * (y * z)), \delta(y)\}$ for all $x, y \in X$.

Definition 2.5. [1] An object of the form $H = \{\langle x, \delta_H(x), \nu_H(x)\rangle : x \in X\}$ is an intuitionistic fuzzy set H in X, where the functions $\delta_H : X \to [0, 1]$ and $\nu_H : X \to [0, 1]$ denote the degree of membership (namely $\delta_H(x)$) and the degree of nonmembership (namely $\nu_H(x)$) of each element $x \in X$ to the set H, respectively, and $0 \leq \delta_H(x) + \nu_H(x) \leq 1$ for each $x \in X$.

Instead of $H = \{\langle x, \delta_H(x), \nu_H(x)\rangle : x \in X\}$ we can use the notation $H = \langle x, \delta_H, \nu_H\rangle$ for our convenience.
The two operators used in this paper are defined as:
If H = (δ_H, ν_H) is an intuitionistic fuzzy set then,

$$\begin{aligned} \triangle H &= \{(x, \delta_H(x), \overline{\delta}_H(x))/x \in X\} \\ \triangledown H &= \{(x, \overline{\nu}_H(x), \nu_H(x))/x \in X\} \end{aligned}$$

For the sake of simplicity, we also use $x \bigcup y$ for $max(x, y)$ and $x \bigcap y$ for $min(x, y)$.

Definition 2.6. [4] An intuitionistic fuzzy set $H = (\delta_H, \nu_H)$ in X is called an intuitionistic fuzzy ideal of X, if it satisfies the following axioms:

(1) $\delta_H(0) \geq \delta_H(x), \nu_H(0) \leq \nu_H(x)$,

(2) $\delta_H(x) \geq \delta_H(x * y) \bigcap \delta_H(y)$,

(3) $\nu_H(x) \leq \nu_H(x * y) \bigcup \nu_H(y)$, for all $x, y \in X$.

Definition 2.7. [7] An intuitionistic fuzzy set $H = (\delta_H, \nu_H)$ in X is called an intuitionistic fuzzy H-ideal of X, if it satisfies the following axioms:

(1) $\delta_H(0) \geq \delta_H(x), \nu_H(0) \leq \nu_H(x)$,

(2) $\delta_H(x * z) \geq \delta_H(x * (y * z)) \bigcap \delta_H(y)$,

(3) $\nu_H(x * z) \leq \nu_H(x * (y * z)) \bigcup \nu_H(y)$, for all $x, y, z \in X$.

Definition 2.8. [3] A fuzzy set $H = \{\langle x, \delta_H(x)\rangle : x \in X\}$ in X is called a doubt fuzzy subalgebra of X if
$\delta_H(x * y) \leq \delta_H(x) \bigcup \delta_H(y)$, for all $x, y \in$ X.

Definition 2.9. [3] A fuzzy set $H = \{\langle x, \delta_H(x)\rangle : x \in X\}$ in X is called a doubt fuzzy ideal of X if

(1) $\delta_H(0) \leq \delta_H(x)$,

(2) $\delta_H(x) \leq \delta_H(x * y) \bigcup \delta_H(y)$, for all $x, y \in X$.

3. Doubt Intuitionistic Fuzzy H-Ideal

In this section, we define doubt intuitionistic fuzzy H-ideals in BH-algebras and investigate its properties.

Definition 3.1. Let $H = (\delta_H, \nu_H)$ be an intuitionistic fuzzy subset of a BH-algebra X, then H is called a *doubt intuitionistic fuzzy H-ideal* of X if

(1) $\delta_H(0) \leq \delta_H(x), \nu_H(0) \geq \nu_H(x)$,

(2) $\delta_H(x * z) \leq \delta_H(x * (y * z)) \bigcup \delta_H(y)$,

(3) $\nu_H(x * z) \geq \nu_H(x * (y * z)) \bigcap \nu_H(y)$, for all $x, y, z \in X$.

Example 3.2. Let $X = \{0, 1, 2\}$ be a BH-algebra with the following cayley table:

$*$	0	1	2
0	0	2	0
1	1	0	1
2	2	1	0

Let $H = (\delta_H, \nu_H)$ be an intuitionistic fuzzy set of X as defined by

X	0	1	2
δ_H	0.3	0.5	0.4
ν_H	0.6	0.1	0.1

Then $H = (\delta_H, \nu_H)$ is a doubt intuitionistic fuzzy H-ideal of X.

Example 3.3. Let $X = \{0, 1, 2, 3\}$ be a BH-algebra with the following cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	2	0
2	2	0	0	1
3	3	2	1	0

Let $H = (\delta_H, \nu_H)$ be an intuitionistic fuzzy set of X as defined by

X	0	1	2	3
δ_H	0.1	0.5	0.8	0.9
ν_H	0.8	0.1	0.1	0.1

Then $H = (\delta_H, \nu_H)$ is a doubt intuitionistic fuzzy H-ideal of X.

Theorem 3.4. Let an intuitionistic fuzzy set $H = (\delta_H, \nu_H)$ in X be a doubt intuitionistic fuzzy H-ideal of an associative BH-algebra X. Then if the inequality $x * p \leq q$ holds in X, then

(1) $\delta_H(x * p) \leq \delta_H(q)$

(2) $\nu_H(x * p) \geq \nu_H(q)$

Proof: Let $x, p, q \in$ X be such that $x * p \leq q$ then $(x * p) * q = 0$ and since H is a doubt intuitionistic fuzzy H-ideal of X, so

$$\begin{aligned}
\delta_H(x * p) &\leq max\{\delta_H(x * (q * p)), \delta_H(q)\}, \\
&= max\{\delta_H((x * q) * p), \delta_H(q)\} \ [since\ X\ is\ associative] \\
&= max\{\delta_H((x * p) * q), \delta_H(q)\} \\
&= max\{\delta_H(0), \delta_H(q)\} \\
&= \delta_H(q) \ [because\ \delta_H(0) \leq \delta_H(q)] \\
\therefore \delta_H(x * p) &\leq \delta_H(q). \ Again \\
\nu_H(x * p) &\geq min\{\nu_H(x * (q * p)), \nu_H(q)\}, \\
&= min\{\nu_H((x * q) * p), \nu_H(q)\} \ [since\ X\ is\ associative] \\
&= min\{\nu_H((x * p) * q), \nu_H(q)\} \\
&= min\{\nu_H(0), \nu_H(q)\} \\
&= \nu_H(q) \ [because\ \nu_H(0) \geq \nu_H(q)] \\
\therefore \nu_H(x * p) &\geq \nu_H(q).
\end{aligned}$$

This completes the proof.

Proposition 3.5. Let an intuitionistic fuzzy set $H = (\delta_H, \nu_H)$ be a doubt intuitionistic fuzzy H-ideal of a BH-algebra X. Then $\delta_H(x*(x*0)) \leq \delta_H(x)$ and $\nu_H(x * (x * 0)) \geq \nu_H(x)$, for all $x \in$ X.

Proof: Let $x \in$ X.

$$\begin{aligned}
\delta_H(x * (x * 0)) &\leq \delta_H(x * (x * (x * 0))) \bigcup \delta_H(x) \\
&= \delta_H(x * (x * x)) \bigcup \delta_H(x) \\
&= \delta_H(x * 0) \bigcup \delta_H(x) \\
&= \delta_H(x) \bigcup \delta_H(x) \\
&= \delta_H(x)
\end{aligned}$$

$\therefore \delta_H(x*(x*0)) \leq \delta_H(x), \forall\, x \in$ X. Again,

$$\begin{aligned} \nu_H(x*(x*0)) &\geq \nu_H(x*(x*(x*0)))\bigcap\delta_H(x) \\ &= \nu_H(x*(x*x))\bigcap\nu_H(x) \\ &= \nu_H(x*0)\bigcap\nu_H(x) \\ &= \nu_H(x)\bigcap\nu_H(x) \\ &= \nu_H(x) \end{aligned}$$

$\therefore \nu_H(x*(x*0)) \geq \nu_H(x), \forall\, x \in$ X.

Lemma 3.6. If an intuitionistic fuzzy set $H = (\delta_H, \nu_H)$ be a doubt intuitionistic fuzzy H-ideal of a BH-algebra X. Then we have the followings, $x \leq p$, then $\delta_H(x) \leq \delta_H(p)$ and $\nu_H(x) \geq \nu_H(p)$, for all $x, p \in$ X.

Proof: Let $x, p \in$ X such that $x \leq p$ then $x * p = 0$. Now,

$$\begin{aligned} \delta_H(x) &= \delta_H(x*0) \\ &\leq max\{\delta_H(x*(p*0)), \delta_H(p)\} \\ &= max\{\delta_H(x*p), \delta_H(p)\} \\ &= max\{\delta_H(0), \delta_H(p)\} \\ &= \delta_H(p). \\ \therefore \delta_H(x) &\leq \delta_H(p). \\ Again,\ \nu_H(x) &= \nu_H(x*0) \\ &\geq min\{\nu_H(x*(p*0)), \nu_H(p)\} \\ &= min\{\nu_H(x*p), \nu_H(p)\} \\ &= min\{\nu_H(0), \nu_H(p)\} \\ &= \nu_H(p). \\ \therefore \nu_H(x) &\geq \nu_H(p). \end{aligned}$$

Theorem 3.7. Let $H = (\delta_H, \nu_H)$ be a doubt intuitionistic fuzzy H-ideal of X. Then so is $\triangle H = \{\langle x, \delta_H(x), \overline{\delta}_H(x)\rangle / x \in X\}$.

Proof: Since $H = (\delta_H, \nu_H)$ is a doubt intuitionistic fuzzy H-ideal of X, then $\delta_H(0) \leq \delta_H(x)$ and $\delta_H(x*z) \leq \delta_H(x*(y*z))\bigcup\delta_H(y)$. Now,

$$\begin{aligned} \delta_H(0) &\leq \delta_H(x), \\ or\ 1-\overline{\delta}_H(0) &\leq 1-\overline{\delta}_H(x), \\ or\ \overline{\delta}_H(0) &\geq \overline{\delta}_H(x),\ for\ any\ x \in X. \end{aligned}$$

Now, for any $x, y, z \in$ X,

$$\begin{aligned} \delta_H(x*z) &\leq max\{\delta_H(x*(y*z)), \delta_H(y)\} \\ \Rightarrow 1-\overline{\delta}_H(x*z) &\leq max\{1-\overline{\delta}_H(x*(y*z)), 1-\overline{\delta}_H(y)\} \\ or\ \overline{\delta}_H(x*z) &\geq 1-max\{1-\overline{\delta}_H(x*(y*z)), 1-\overline{\delta}_H(y)\} \\ \therefore \overline{\delta}_H(x*z) &\geq min\{\overline{\delta}_H(x*(y*z)), \overline{\delta}_H(y)\}. \end{aligned}$$

Hence $\triangle H = \{\langle x, \delta_H(x), \overline{\delta}_H(x)\rangle / x \in X\}$ is a doubt intuitionistic fuzzy H-ideal of X.

Theorem 3.8. Let $H = (\delta_H, \nu_H)$ be a doubt intuitionistic fuzzy H-ideal of X. Then so is $\bigtriangledown H = \{\langle x, \overline{\nu}_H(x), \nu_H(x)\rangle / x \in X\}$.
Proof: Since $H = (\delta_H, \nu_H)$ is a doubt intuitionistic fuzzy H-ideal of X, then $\nu_H(0) \geq \nu_H(x)$ and $\nu_H(x*z) \geq \nu_H(x*(y*z)) \bigcap \nu_H(y)$. Now,

$$\begin{aligned} \nu_H(0) &\geq \nu_H(x), \\ or\ 1-\overline{\nu}_H(0) &\geq 1-\overline{\nu}_H(x), \\ or\ \overline{\nu}_H(0) &\leq \overline{\nu}_H(x),\ for\ any\ x \in X. \end{aligned}$$

Now, for any $x, y, z \in$ X,

$$\begin{aligned} \nu_H(x*z) &\geq min\{\nu_H(x*(y*z)), \nu_H(y)\} \\ \Rightarrow 1-\overline{\nu}_H(x*z) &\geq min\{1-\overline{\nu}_H(x*(y*z)), 1-\overline{\nu}_H(y)\} \\ or\ \overline{\nu}_H(x*z) &\leq 1-min\{1-\overline{\nu}_H(x*(y*z)), 1-\overline{\nu}_H(y)\} \\ \therefore \overline{\nu}_H(x*z) &\leq max\{\overline{\nu}_H(x*(y*z)), \overline{\nu}_H(y)\}. \end{aligned}$$

Hence $\bigtriangledown H = \{\langle x, \overline{\nu}_H(x), \nu_H(x)\rangle / x \in X\}$ is a doubt intuitionistic fuzzy H-ideal of X.

Theorem 3.9. Let $H = (\delta_H, \nu_H)$ be an intuitionistic fuzzy set in X. Then $H = (\delta_H, \nu_H)$ is a doubt intuitionistic fuzzy H-ideal of X if and only if $\triangle H = \{\langle x, \delta_H(x), \overline{\delta}_H(x)\rangle / x \in X\}$ and $\bigtriangledown H = \{\langle x, \overline{\nu}_H(x), \nu_H(x)\rangle / x \in X\}$ are doubt intuitionistic fuzzy H-ideals of X.
Proof: The proof is same as Theorem 3.7 and Theorem 3.8.
Let us illustrate the Theorem 3.7, Theorem 3.8 and Theorem 3.9 using the following example.

Example 3.10. Let us consider the Example 3.2, $\triangle H = \{\langle x, \delta_H(x), \overline{\delta}_H(x)\rangle / x \in X\}$ the values of $\delta_H(x)$ and $\overline{\delta}_H(x)$ are defined as follows:

X	0	1	2
δ_H	0.3	0.5	0.4
$\overline{\delta}_H$	0.7	0.4	0.4

Also, $\bigtriangledown H = \{\langle x, \overline{\nu}_H(x), \nu_H(x)\rangle / x \in X\}$ whose $\nu_H(x)$ and $\overline{\nu}_H(x)$ are defined by

X	0	1	2
$\overline{\nu}_H$	0.2	0.4	0.3
ν_H	0.6	0.1	0.1

So, it can be verified that $\triangle H = \{\langle x, \delta_H(x), \overline{\delta}_H(x)\rangle / x \in X\}$ and $\bigtriangledown H = \{\langle x, \overline{\nu}_H(x), \nu_H(x)\rangle / x \in X\}$ are doubt intuitionistic fuzzy H-ideals of X.

Example 3.11. Let us consider the Example 3.3, $\triangle H = \{\langle x, \delta_H(x), \overline{\delta}_H(x)\rangle / x \in X\}$ the values of $\delta_H(x)$ and $\overline{\delta}_H(x)$ are defined as follows:

X	0	1	2	3
δ_H	0.1	0.5	0.8	0.9
$\overline{\delta}_H$	0.7	0.5	0.5	0.5

Also, $\bigtriangledown H = \{\langle x, \overline{\nu}_H(x), \nu_H(x)\rangle / x \in X\}$ whose $\nu_H(x)$ and $\overline{\nu}_H(x)$ are defined by

X	0	1	2	3
$\overline{\nu}_H$	0.2	0.4	0.7	0.7
ν_H	0.8	0.1	0.1	0.1

So, it can be verified that $\triangle H = \{\langle x, \delta_H(x), \overline{\delta}_H(x)\rangle / x \in X\}$ and $\bigtriangledown H = \{\langle x, \overline{\nu}_H(x), \nu_H(x)\rangle / x \in X\}$ are doubt intuitionistic fuzzy H-ideals of X.

Theorem 3.12. An intuitionistic fuzzy set $H = (\delta_H, \nu_H)$ is a doubt intuitionistic fuzzy H-ideal of a BH-algebra X if and only if the fuzzy sets δ_H and $\overline{\nu}_H$ are doubt fuzzy H-ideals of X.

Proof: Let $H = (\delta_H, \nu_H)$ be a doubt intuitionistic fuzzy H-ideal of a BH-algebra X. Then it is obvious that δ_H is a doubt fuzzy H-ideal of X, and from Theorem 3.9, we can prove that $\overline{\nu}_H$ is a doubt fuzzy H-ideal of X. Conversely, let δ_H be a doubt fuzzy H-ideal of X. Therefore, $\delta_H(0) \leq \delta_H(x)$ and $\delta_H(x) \leq max\{\delta_H(x * (y * z)), \delta_H(y)\}$, for all $x, y, z \in$ X. Again, since

$\overline{\nu}_H$ is a doubt fuzzy H-ideal of X, so

$$\begin{aligned}\overline{\nu}_H(0) &\leq \overline{\nu}_H(x)\\ \Rightarrow 1-\nu_H(0) &\leq 1-\nu_H(x)\\ \Rightarrow \nu_H(0) &\geq \nu_H(x)\\ Also,\ \overline{\nu}_H(x*z) &\leq max\{\overline{\nu}_H(x*(y*z)),\overline{\nu}_H(y)\}\\ or\ 1-\nu_H(x*z) &\leq max\{1-\nu_H(x*(y*z)),1-\nu_H(y)\}\\ or\ \nu_H(x*z) &\geq 1-max\{1-\nu_H(x*(y*z)),1-\nu_H(y)\}\\ Finally,\ \nu_H(x*z) &\geq min\{\nu_H(x*(y*z)),\nu_H(y)\},\forall\ x,y,z\in X.\end{aligned}$$

Hence, $H=(\delta_H,\nu_H)$ is a doubt intuitionistic fuzzy H-ideal of X.

Corollary 3.13. Let $H=(\delta_H,\nu_H)$ be a doubt intuitionistic fuzzy H-ideal of a BH-algebra X. Then the sets, $N_{\delta_H}=\{x\in X/\delta_H(x)=\delta_H(0)\}$, and $N_{\nu_H}=\{x\in X/\nu_H(x)=\nu_H(0)\}$ are H-ideals of X.

Proof: Let $H=(\delta_H,\nu_H)$ be a doubt intuitionistic fuzzy H-ideal of a BH-algebra X. Obviously, $0\in N_{\delta_H}$ and N_{ν_H}. Now, let $x,y,z\in$ X, such that $x*(y*z),y\in N_{\delta_H}$. Then $\delta_H(x*(y*z))=\delta_H(0)=\delta_H(y)$. Now, $\delta_H(x*(y*z))\leq max\{\delta_H(x*(y*z)),\delta_H(y)\}=\delta_H(0)$.

Again, since δ_H is a doubt fuzzy H-ideal of X, $\delta_H(0)\leq\delta_H(x*z)$. Therefore, $\delta_H(0)=\delta_H(x*z)$. It follows that, $x*z\in N_{\delta_H}$, for all $x,y,z\in$ X. Therefore, N_{δ_H} is an H-ideal of X. Following the same way we can prove that N_{ν_H} is also an H-ideal of X.

Definition 3.14. Let $H=(\delta_H,\nu_H)$ be an intuitionistic fuzzy set of X, and $i,j\in$ [0,1], then δ level i-cut and ν level j-cut of H, is as follows:

$$\begin{aligned}\delta^{\leq}_{H,i} &= \{x\in X/\delta_H(x)\leq i\}\\ and\ \nu^{\geq}_{H,j} &= \{x\in X/\nu_H(x)\geq j\}.\end{aligned}$$

Theorem 3.15. If $H=(\delta_H,\nu_H)$ be a doubt intuitionistic fuzzy H-ideal of X, then $\delta^{\leq}_{H,i}$ and $\nu^{\geq}_{H,j}$ are H-ideals of X for any $i,j\in$ [0,1].

Proof: Let $H=(\delta_H,\nu_H)$ be a doubt intuitionistic fuzzy H-ideal of X, and let $i\in$ [0,1] with $\delta_H(0)\leq i$. Then we have, $\delta_H(0)\leq\delta_H(x)$, for all $x\in$ X, but $\delta_H(x)\leq i$, for all $x\in\delta^{\leq}_{H,i}$. So, $0\in\delta^{\leq}_{H,i}$. Let $x,y,z\in$ X be such that $x*(y*z))\in\delta^{\leq}_{H,i}$ and $y\in\delta^{\leq}_{H,i}$, then, $\delta_H(x*(y*z))\in\delta^{\leq}_{H,i}$ and $\delta_H(y)\in\delta^{\leq}_{H,i}$. Therefore, $\delta_H(x*(y*z))\leq i$ and $\delta_H(y)\leq i$. Since δ_H is a doubt fuzzy H-ideal of X, it follows that, $\delta_H(x*z)\leq\delta_H((x*(y*z))\bigcup\delta_H(y)\leq i$ and hence $x*z\in\delta^{\leq}_{H,i}$, for all $x,y,z\in$ X. Therefore, $\delta^{\leq}_{H,i}$ is an H-ideal of X for $i \in$ [0,1]. Similarly, we can prove that $\nu^{\geq}_{H,j}$ is an H-ideal of X for $j \in$ [0,1].

Theorem 3.16. If $\delta^{\leq}_{H,i}$ and $\nu^{\geq}_{H,j}$ are either empty or H-ideals of X for $i, j \in$ [0,1], then $H = (\delta_H, \nu_H)$ is a doubt intuitionistic fuzzy H-ideal of X.
Proof: Let $\delta^{\leq}_{H,i}$ and $\nu^{\geq}_{H,j}$ be either empty or H-ideals of X for $i, j \in$ [0,1]. For any $x \in$ X, let $\delta_H(x) = i$ and $\nu_H(x) = j$. Then $x \in \delta^{\leq}_{H,i} \bigcap \nu^{\geq}_{H,j}$, so $\delta^{\leq}_{H,i} \neq \phi \neq \nu^{\geq}_{H,j}$. Since $\delta^{\leq}_{H,i}$ and $\nu^{\geq}_{H,j}$ are H-ideals of X, therefore $0 \in \delta^{\leq}_{H,i} \bigcap \nu^{\geq}_{H,j}$. Hence $\delta_H(0) \leq i = \delta_H(x)$ and $\nu_H(0) \geq j = \nu_H(x)$, where $x \in$ X. If there exist $x', y', z' \in$ X such that $\delta_H(x' * z') > max\{\delta_H(x' * (y' * z')), \delta_H(y')\}$, then by taking, $i_0 = \frac{1}{2}(\delta_H(x' * z') + max\{\delta_H(x' * (y' * z')), \delta_H(y')\})$, we have $\delta_H(x' * z') > i_0 > max\{\delta_H(x' * (y' * z')), \delta_H(y')\}$. Hence, $x' * z' \notin \delta^{\leq}_{H,i_0}, (x' * (y' * z')) \in \delta^{\leq}_{H,i_0}$ and $y' \in \delta^{\leq}_{H,i_0}$, that is $\delta^{\leq}_{H,i_0}$ is not an H-ideal of X, which is a contradiction. Therefore, $\delta_H(x * z) \leq \delta_H((x * (y * z)) \bigcup \delta_H(y)$, for any $x, y, z \in$ X.
Finally, assume that there exist $a, b, c \in$ X such that $\nu_H(a * c) < min\{\nu_H(a * (b * c)), \nu_H(b)\}$. Taking $j_0 = \frac{1}{2}(\nu_H(a * c) + min\{\nu_H(a * (b * c)), \nu_H(b)\})$, then $min\{\nu_H(a * (b * c)), \nu_H(b)\} > j_0 > \nu_H(a * c)$. Therefore, $a * (b * c) \in \nu^{\geq}_{H,j}$ and $b \in \nu^{\geq}_{H,j}$ but $a * c \notin \nu^{\geq}_{H,j}$. Again a contradiction. This completes the proof.

But, if an intuitionistic fuzzy set $H = (\delta_H, \nu_H)$ is not a doubt intuitionistic fuzzy H-ideal of X, then $\delta^{\leq}_{H,i}$ and $\nu^{\geq}_{H,j}$ are not H-ideals of X for $i, j \in$ [0,1], which is illustrated in the following example.

Example 3.17. Let $X = \{0, 1, 2, 3\}$ be a BH-algebra with the following cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Let $H = (\delta_H, \nu_H)$ be an intuitionistic fuzzy set of X as defined by

X	0	1	2	3
δ_H	0.1	0.8	0.4	0.3
ν_H	0.9	0.2	0.4	0.5

which is not a doubt intuitionistic fuzzy H-ideal of X.
For i = 0.86 and j = 0.18, we get $\delta^{\leq}_{H,i} = \nu^{\geq}_{H,j} = \{0,1,2\}$, which are not H-ideals of X, as $1*(3*2) = 1*1 = 0 \in \{0,1,2\}$, and $0 \in \{0,1,2\}$, but $1*2 = 3 \notin \{0,1,2\}$.

Theorem 3.18. Union of any two doubt intuitionistic fuzzy H-ideals of X, is also a doubt intuitionistic fuzzy H-ideal of X.
Proof: Let $H = (\delta_H, \nu_H)$ and $I = (\delta_I, \nu_I)$ be two doubt intuitionistic fuzzy H-ideals of X. Again let, $M = H \cup I = (\delta_M, \nu_M)$, where $\delta_M = \delta_H \cup \delta_I$ and $\nu_M = \delta_H \cap \delta_I$. Let $x \in$ X, then $\delta_M(0) = (\delta_H \cup \delta_I)(0) = max\{\delta_H(0), \delta_I(0)\} \leq max\{\delta_H(x), \delta_I(x)\} = (\delta_H \cup \delta_I)(x) = \delta_M(x)$ and $\nu_M(0) = (\nu_H \cap \nu_I)(0) = min\{\nu_H(0), \nu_I(0)\} \geq min\{\nu_H(x), \nu_I(x)\} = (\nu_H \cap \nu_I)(x) = \nu_M(x)$. Also,

$$\begin{aligned}
\delta_M(x*z) &= max\{\delta_H(x*z), \delta_I(x*z)\} \\
&\leq max\{max[\delta_H(x*(y*z)), \delta_H(y)], max[\delta_I(x*(y*z)), \delta_I(y)]\} \\
&= max\{max[\delta_H(x*(y*z)), \delta_I(x*(y*z))], max[\delta_H(y), \delta_I(y)]\} \\
&= max[\delta_M(x*(y*z)), \delta_M(y)].
\end{aligned}$$

Similarly, we can prove that, $\nu_M(x*z) \geq min[\nu_M(x*(y*z)), \nu_M(y)]$. This completes the proof.

Theorem 3.19. Let H and I be two intuitionistic fuzzy subsets of X, such that one is contained another. Also H and I are two doubt intuitionistic fuzzy H-ideals of X. Then the intersection of H and I are also doubt intuitionistic fuzzy H-ideal of X.
Proof: Let $H = (\delta_H, \nu_H)$ and $I = (\delta_I, \nu_I)$ be two doubt intuitionistic fuzzy H-ideals of X. Again let, $N = H \cap I = (\delta_N, \nu_N)$, where $\delta_N = \delta_H \cap \delta_I$ and $\nu_N = \delta_H \cup \delta_I$. Let $x \in$ X, then $\delta_N(0) = \delta_H(0) \cap \delta_I(0) \leq \delta_H(x) \cap \delta_I(x) = \delta_N(x)$ and $\nu_N(0) = \nu_H(0) \cup \nu_I(0) \geq \nu_H(x) \cup \nu_I(x) = \nu_N(x)$. Also,

$$\begin{aligned}
\delta_N(x*z) &= \delta_H(x*z) \cap \delta_I(x*z) \\
&\leq max[\delta_H(x*(y*z)), \delta_H(y)] \cap max[\delta_I(x*(y*z)), \delta_I(y)] \\
&= max\{[\delta_H(x*(y*z)) \cap \delta_I(x*(y*z))], [\delta_H(y) \cap \delta_I(y)]\}, \\
&\quad [because\ one\ is\ contained\ another] \\
&= max[\delta_N(x*(y*z)), \delta_N(y)].
\end{aligned}$$

Similarly, we can prove that, $\nu_N(x*z) \geq min[\nu_N(x*(y*z)), \nu_N(y)]$. This completes the proof.
Theorem 3.18 and Theorem 3.19 are verified by the following example.

Example 3.20. Let $X = \{0, 1, 2\}$ be a BH-algebra with the following cayley table:

$*$	0	1	2
0	0	1	0
1	1	0	1
2	2	1	0

Let $H = (\delta_H, \nu_H)$ be an intuitionistic fuzzy set of X as defined by

X	0	1	2
δ_H	0.2	0.3	0.3
ν_H	0.7	0.6	0.6

Then $H = (\delta_H, \nu_H)$ is a doubt intuitionistic fuzzy H-ideal of X. Again, let $I = (\delta_I, \nu_I)$ be an intuitionistic fuzzy set of X as defined by

X	0	1	2
δ_I	0.1	0.4	0.4
ν_I	0.8	0.3	0.3

Then $I = (\delta_I, \nu_I)$ is a doubt intuitionistic fuzzy H-ideal of X. We also assume that $F = H \cup I = (\delta_F, \nu_F)$ where $\delta_F = \delta_H \cup \delta_I$ and $\nu_F = \nu_H \cap \nu_I$ and F is defined as:

X	0	1	2
δ_F	0.2	0.4	0.4
ν_F	0.8	0.6	0.6

Then $F = (\delta_F, \nu_F)$ is a doubt intuitionistic fuzzy H-ideal of X. Now let, $G = H \cap I = (\delta_H, \nu_G)$ where $\delta_G = \delta_H \cap \delta_I$ and $\nu_G = \nu_H \cup \nu_I$. Then G is an intuitionistic fuzzy set of X which can be defined as:

X	0	1	2
δ_G	0.1	0.3	0.3
ν_G	0.7	0.3	0.3

Then it is clear that $G = (\delta_G, \nu_G)$ is a doubt intuitionistic fuzzy H-ideal of X.

Theorem 3.21. Every doubt intuitionistic fuzzy H-ideal of X is a doubt intuitionistic fuzzy ideal of X.

Proof: Let $H = (\delta_H, \nu_H)$ be a doubt intuitionistic fuzzy H-ideal of X, then (i) $\delta_H(0) \leq \delta_H(x); \nu_H(0) \geq \nu_H(x)$, (ii) $\delta_H(x * z) \leq \delta_H(x * (y * z)) \bigcup \delta_H(y)$, and (iii) $\nu_H(x * z) \geq \nu_H(x * (y * z)) \bigcap \nu_H(y)$, for all $x, y, z \in$ X.

If we put z = 0, then from (ii) and (iii), we get $\delta_H(x) \leq \delta_H(x*y) \bigcup \delta_H(y)$ and $\nu_H(x) \geq \nu_H(x*y) \bigcap \nu_H(y)$, for all $x, y, z \in$ X, since $x*0 = x$, for all $x \in$ X. Hence, H is a doubt intuitionistic fuzzy ideal of X.

But the converse may not be true. That is every doubt intuitionistic fuzzy ideal of X is not a doubt intuitionistic fuzzy H-ideal of X. It can be verified by the following example.

Example 3.22. Let $X = \{0, 1, 2\}$ be a BH-algebra with the following cayley table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	0	0

Let $H = (\delta_H, \nu_H)$ be an intuitionistic fuzzy set of X as defined by

X	0	1	2
δ_H	0.4	0.8	0.5
ν_H	1	0.3	0.2

Then $H = (\delta_H, \nu_H)$ is a doubt intuitionistic fuzzy ideal of X. But H is not a doubt intuitionistic fuzzy H-ideal of X, as $\delta_H(1*2) \nleq max\{\delta_H(1*(0*2)), \delta_H(0)\}$. Because, $\delta_H(1*2) = 0.5$ and $max\{\delta_H(1*(0*2)), \delta_H(0)\} = \delta_H(0) = 0.4$.

References

[1] ATANASSOV, K.T.: Intuitionistic fuzzy sets, Fuzzy sets and Systems, 35 (1986), 87-96.

[2] ATANASSOV, K.T.: New operations defined over the intuitionistic fuzzy sets, Fuzzy sets and Systems, 61(1994),137-142.

[3] JUN, Y. B.: Doubt fuzzy BCK/BCI-algebras, Soochow J.Math.Stat. 20(3) (1994) 351-358.

[4] JUN, Y. B. and KIM, K. H.: Intuitionistic fuzzy ideals of BCK-algebras, Int. J. Math.Sci. 24 (2000) 839-849.

[5] JUN, Y. B., ROH, E. H. and KIM, H. S.: On BH-algebras, Sci. Mathematicae 1 (1998), 347-354.

[6] PARK, C. H.: Interval-valued fuzzy ideal in BH-algebras, Advance in fuzzy set and system 1 (2006), 231-240.

[7] SATYANARAYAN, B.,MADHAVI, U. B. and PRASAD, R.D.: On foldness of intuitionistic fuzzy H-ideals in BCK-algebras, Int. Math. Forum 5 (45-48) (2010) 2205-2211.

[8] ZADEH, L. A.: Fuzzy sets, Inform. Control 8 (1965), 338-353.