2-DOMINATING SETS AND 2- DOMINATION POLYNOMIAL OF PATHS

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Abstract: *Let G be a simple connected graph of order n. Let* $D_2(G, i)$ *be the family of 2-dominating sets in G with cardinality i. The polynomial* $D_2(G, x) = \sum_{i=\gamma_2(G)}^{n} d_2(G,i)x^i$ *is called the 2-domination polynomial of G. In this paper we obtain a recursive formula for* $d_2(P_n,i)$. *Using this recursive formula we construct the 2-domination polynomial,* $D_2(P_n,x) = \sum_{i=\lceil\frac{n+1}{2}\rceil}^{n} d_2(P_n,i)x^i$, *where* $d_2(P_n, i)$ *is the number of 2-dominating sets of* P_n *of cardinality i and some properties of this polynomial have been studied.*

Keywords: **Path, 2-dominating set, 2-domination number, 2-domination polynomial.**

1. INTRODUCTION

Let G = (V, E) be a simple graph of order n. For any vertex v∈V, the open neighbourhood of V is the set N(v) = {u∈V / uv∈E} and the closed neighbourhood of V is the set N[v] = N(v) ∪ {v}. For a set S ⊆V, the open neighbourhood of S is N(S) = $\bigcup_{v\in S} N(v)$ and the closed neighbourhood of S is N[S] = N(S) ∪ S.

A set D ⊆V is a dominating set of G if N[D] = V or equivalently, every vertex in V – D is adjacent to at least one vertex in D.

The domination number of a graph G is defined as the minimum cardinality taken over all dominating sets D of vertices in G and is denoted by γ(G).

We use the notation $\lceil x\rceil$ for the smallest integer greater than or equal to x and $\lfloor x\rfloor$ for the largest integer less than or equal to x. Also, we denote the set {1,2,3,…,n} by [n], throughout this paper.

2. 2 - DOMINATING SETS OF PATHS

In this section, we state the 2-domination number of path and some of its properties.

Definition 2.1:

Let G be a simple graph of order n with no isolated vertices. A subset D ⊆ V is a 2-dominating set of the graph G, if every vertex v∈ V – D is adjacent to at least 2 vertices in D. The 2-domination number $\gamma_2(G)$ is the minimum cardinality among the 2-dominating sets of G.

Lemma 2.2:

Let P_n be the path with n vertices, then its 2-domination number is $\gamma_2(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$.

Lemma 2.3:

Let P_n, $n \geq 4$ be the path with $|V(P_n)| = n$. Then, $d_2(P_n, i) = 0$ if $i < \left\lceil \frac{n+1}{2} \right\rceil$ or $i > n$ and $d_2(P_n, i) > 0$ if $\left\lceil \frac{n+1}{2} \right\rceil \leq i \leq n$.

Proof:

If $i < \left\lceil \frac{n+1}{2} \right\rceil$ or $i > n$, then there is no 2-dominating set of cardinality i. Therefore, $D_2(P_n,i) = \phi$. By Lemma 2.2, the cardinality of the minimum 2-dominating set is $\left\lceil \frac{n+1}{2} \right\rceil$. Therefore, $d_2(P_n, i) > 0$ if $i \geq \left\lceil \frac{n+1}{2} \right\rceil$ and $i \leq n$. Hence, we have, $d_2(P_n, i) = 0$ if $i < \left\lceil \frac{n+1}{2} \right\rceil$ or $i > n$ and $d_2(P_n, i) > 0$ if $\left\lceil \frac{n+1}{2} \right\rceil \leq i \leq n$.

Lemma 2.4:

Let P_n, n≥ 4 be the path with $|V(P_n)| = n$.

(i) If $D_2(P_{n-1}, i-1) = \phi$ and $D_2(P_{n-3}, i-1) = \phi$, then $D_2(P_{n-2}, i-1) = \phi$.

(ii) If $D_2(P_{n-1}, i-1) \neq \phi$ and $D_2(P_{n-3}, i-1) \neq \phi$, then $D_2(P_{n-2}, i-1) \neq \phi$.

(iii) If $D_2(P_{n-1}, i-1) = \phi$ and $D_2(P_{n-2}, i-1) = \phi$, then $D_2(P_n, i) = \phi$.

(iv) If $D_2(P_{n-1}, i-1) \neq \phi$ and $D_2(P_{n-2}, i-1) \neq \phi$ then $D_2(P_n, i) \neq \phi$.

Proof:

(i) Since $D_2(P_{n-1}, i-1) = \phi$ and $D_2(P_{n-3}, i-1) = \phi$,
by Lemma 2.3,
$i-1 > n-1$ or $i-1 < \left\lceil \frac{n}{2} \right\rceil$ and $i-1 > n-3$ or $i-1 < \left\lceil \frac{n-2}{2} \right\rceil$.
Therefore, $i-1 > n-1$ or $i-1 < \left\lceil \frac{n-2}{2} \right\rceil$.
Hence, $i-1 > n-2$ or $i-1 < \left\lceil \frac{n-1}{2} \right\rceil$ holds. Therefore, $D_2(P_{n-2}, i-1) = \phi$.

(ii) Suppose that $D_2(P_{n-2}, i-1) = \phi$, So by Lemma 2.3,
we have, $i-1 > n-2$ or $i-1 < \left\lceil \frac{n-1}{2} \right\rceil$. If $i-1 > n-2$, then $i-1 > n-3$.
Therefore, $D_2(P_{n-3}, i-1) = \phi$, a contradiction. If $i-1 < \left\lceil \frac{n-1}{2} \right\rceil$, then $i-1 < \left\lceil \frac{n}{2} \right\rceil$.
Therefore, $D_2(P_{n-1}, i-1) = \phi$, a contradiction. Hence, $D_2(P_{n-2}, i-1) \neq \phi$.

(iii) Since $D_2(P_{n-1}, i-1) = \phi$ and $D_2(P_{n-2}, i-1) = \phi$, By Lemma 2.3,
$i-1 > n-1$ or $i-1 < \left\lceil \frac{n}{2} \right\rceil$ and $i-1 > n-2$ or $i-1 < \left\lceil \frac{n-1}{2} \right\rceil$.
Therefore, $i-1 > n-1$ or $i-1 < \left\lceil \frac{n-1}{2} \right\rceil$. Hence, $i > n$ or $i < \left\lceil \frac{n+1}{2} \right\rceil$ holds.
Therefore, $D_2(P_n, i) = \phi$.

(iv) By hypothesis, $\left\lceil\frac{n}{2}\right\rceil \leq i-1 \leq n-1$ and $\left\lceil\frac{n-1}{2}\right\rceil \leq i-1 \leq n-2$.
Therefore, $\left\lceil\frac{n-1}{2}\right\rceil \leq i-1 \leq n-1$. Therefore, $\left\lceil\frac{n+1}{2}\right\rceil \leq i \leq n$ holds.
Therefore, $D_2(P_n, i) \neq \phi$.

Lemma 2.5:

Suppose that $D_2(P_n, i) \neq \phi$, then for every $n \geq 5$, we have

(i) $D_2(P_{n-1}, i-1) = \phi$, $D_2(P_{n-2}, i-1) \neq \phi$, $D_2(P_{n-3}, i-1) \neq \phi$ if and only if $n=2k-1$ and $i = k$ for some $k \geq 3$.
(ii) $D_2(P_{n-2}, i-1) = \phi$, $D_2(P_{n-3}, i-1) = \phi$ and $D_2(P_{n-1}, i-1) \neq \phi$ if and only if $i = n$.
(iii) $D_2(P_{n-1}, i-1) \neq \phi$, $D_2(P_{n-2}, i-1) \neq \phi$ and $D_2(P_{n-3}, i-1) = \phi$ if and only if $i = n-1$.
(iv) $D_2(P_{n-1}, i-1) \neq \phi$, $D_2(P_{n-2}, i-1) \neq \phi$, $D_2(P_{n-3}, i-1) \neq \phi$ iff $\left\lceil\frac{n}{2}\right\rceil + 1 \leq i \leq n-2$.

Proof:

(i) Assume that $D_2(P_{n-1}, i-1) = \phi$, $D_2(P_{n-2}, i-1) \neq \phi$ and $D_2(P_{n-3}, i-1) \neq \phi$
Since, $D_2(P_{n-1}, i-1) = \phi$, by Lemma 2.3, $i-1 > n-1$ or $i-1 < \left\lceil\frac{n}{2}\right\rceil$.
If $i-1 > n-1$, then $i > n$, which implies $D_2(P_n, i) = \emptyset$, which is a contradiction.
Therefore, $i-1 < \left\lceil\frac{n}{2}\right\rceil$. That is, $i \leq \left\lceil\frac{n}{2}\right\rceil$ -------------------(1)
Since $D_2(P_{n-2}, i-1) \neq \phi$, we have $\left\lceil\frac{n-1}{2}\right\rceil \leq i-1 \leq n-2$.
Therefore, $\left\lceil\frac{n-1}{2}\right\rceil + 1 \leq i$ ------------------(2). Since, $D_2(P_{n-3}, i-1) \neq \phi$, we have
$\left\lceil\frac{n-2}{2}\right\rceil \leq i-1 \leq n-3$. Therefore, $\left\lceil\frac{n-2}{2}\right\rceil + 1 \leq i$ -------------(3)
Since, $D_2(P_n, i) \neq \phi$, we have $\left\lceil\frac{n+1}{2}\right\rceil \leq i \leq n$, Therefore, $\left\lceil\frac{n+1}{2}\right\rceil \leq i$ -------------(4)
Combining all these inequalities, we have $\left\lceil\frac{n+1}{2}\right\rceil \leq i \leq \left\lceil\frac{n}{2}\right\rceil$ --------------------(5)
when $n \neq 2k-1$, we get an inequality of the form $s \leq i < s$, Which is not possible.
when $n = 2k-1$, (5) holds . In this case $i = k$.
Conversely, assume that $n = 2k-1$ and $i = k$. Therefore, $n-1 = 2k-2$ and $i-1 = k-1$.
Therefore, $k = \frac{n+1}{2}$ and $k-1 = \frac{n-1}{2}$. We have, $i-1 = k-1 = \frac{n-1}{2} < \left\lceil\frac{n}{2}\right\rceil$
Therefore, $D_2(P_{n-1}, i-1) = \phi$. Also, $D_2(P_{n-2}, i-1) = D_2(P_{2k-3}, k-1) \neq \phi$, since
$\left\lceil\frac{2k-3+1}{2}\right\rceil = \left\lceil\frac{2k-2}{2}\right\rceil = k-1$. $D_2(P_{n-3}, i-1) = D_2(P_{2k-4}, k-1) \neq \phi$. Since
$\left\lceil\frac{2k-4+1}{2}\right\rceil = \left\lceil\frac{2k-3}{2}\right\rceil = k-1$.

(ii) Assume that $D_2(P_{n-2}, i-1) = \phi$, $D_2(P_{n-3}, i-1) = \phi$ and $D_2(P_{n-1}, i-1) \neq \phi$.
Since, $D_2(P_{n-2}, i-1) = \phi$ and $D_2(P_{n-3}, i-1) = \phi$, by Lemma 2.3, we have,
$i-1 > n-2$ or $i-1 < \left\lceil\frac{n-1}{2}\right\rceil$ and $i-1 > n-3$ or $i-1 < \left\lceil\frac{n-2}{2}\right\rceil$.
Therefore, $i-1 > n-2$ or $i-1 < \left\lceil\frac{n-2}{2}\right\rceil$. Since $D_2(P_{n-1}, i-1) \neq \phi$,
we have $\left\lceil\frac{n}{2}\right\rceil \leq i-1 \leq n-1$. If $i-1 < \left\lceil\frac{n-2}{2}\right\rceil$, then $i-1 < \left\lceil\frac{n}{2}\right\rceil$.
Therefore, by Lemma 2.3, $D_2(P_{n-1}, i-1) \neq \phi$, which is a contradiction.
So we have, $i-1 > n-2$. That is, $i > n-1$. Therefore, $i \geq n$ -----------------(1).
Also, Since $D_2(P_n, i) \neq \phi$, $i \leq n$ -----------------(2). Combining these we get $i = n$.
Conversely, if $i = n$,
$D_2(P_{n-2}, i-1) = D_2(P_{n-2}, n-1) = \phi$

$D_2(P_{n-3}, i-1) = D_2(P_{n-3}, n-1) = \phi$
$D_2(P_{n-1}, i-1) = D_2(P_{n-1}, n-1) \neq \phi$

(iii) Assume that $D_2(P_{n-1}, i-1) \neq \phi$, $D_2(P_{n-2}, i-1) \neq \phi$ and $D_2(P_{n-3}, i-1) = \phi$.
Since, $D_2(P_{n-3}, i-1) = \phi$ then by Lemma 2.3, $i-1 > n-3$ or $i-1 < \left\lceil \frac{n-2}{2} \right\rceil$ --------------(1)
Since, $D_2(P_{n-1}, i-1) \neq \phi$, we have $\left\lceil \frac{n}{2} \right\rceil \leq i-1 \leq n-1$ --------------(2)
Suppose, $i-1 < \left\lceil \frac{n-2}{2} \right\rceil$ then (2) does not hold. Therefore our assumption is wrong.
Therefore, $i-1 > n-3$. Also, since $D_2(P_{n-2}, i-1) \neq \phi$, $\left\lceil \frac{n-1}{2} \right\rceil \leq i-1 \leq n-2$ ----- (3)
But, $i-1 > n-3$. Therefore, $i-1 \geq n-2$ -------------------- (4)
From (3) and (4) we get, $i-1 = n-2$. Therefore, $i = n-1$.
Conversely, Suppose $i = n-1$.
Then, $D_2(P_{n-1}, i-1) = D_2(P_{n-1}, n-2) \neq \phi$,
$D_2(P_{n-2}, i-1) = D_2(P_{n-2}, n-2) \neq \phi$ and $D_2(P_{n-3}, i-1) = D_2(P_{n-3}, n-2) = \phi$.

(iv) Assume that, $D_2(P_{n-1}, i-1) \neq \phi$, $D_2(P_{n-2}, i-1) \neq \phi$, $D_2(P_{n-3}, i-1) \neq \phi$,
Then by Lemma 2.3, We have, $\left\lceil \frac{n}{2} \right\rceil \leq i-1 \leq n-1$, $\left\lceil \frac{n-1}{2} \right\rceil \leq i-1 \leq n-2$ and
$\left\lceil \frac{n-2}{2} \right\rceil \leq i-1 \leq n-3$. So, $\left\lceil \frac{n}{2} \right\rceil \leq i-1 \leq n-3$ and hence $\left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n-2$.
Conversely, Suppose $\left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n-2$.
Therefore, $\left\lceil \frac{n}{2} \right\rceil \leq i-1 \leq n-3$. Then,
$\left\lceil \frac{n-1}{2} \right\rceil \leq i-1 \leq n-2$, $\left\lceil \frac{n-2}{2} \right\rceil \leq i-1 \leq n-3$, $\left\lceil \frac{n}{2} \right\rceil \leq i-1 \leq n-1$ holds.
From these we obtain, $D_2(P_{n-2}, i-1) \neq \phi$, $D_2(P_{n-3}, i-1) \neq \phi$ and $D_2(P_{n-1}, i-1) \neq \phi$.

Theorem 2.6

For every $n \geq 3$ and $i > \left\lceil \frac{n+1}{2} \right\rceil$
(i) $D_2(P_{2n-1}, n) = \{1, 3, 5, 7 \ldots .2n-1\}$.
(ii) If $D_2(P_{n-2}, i-1) = \phi$, $D_2(P_{n-3}, i-1) = \phi$ and $D_2(P_{n-1}, i-1) \neq \phi$,
then $D_2(P_n, i) = D_2(P_n, n) = [n]$.
(iii) If $D_2(P_{n-1}, i-1) \neq \phi$, $D_2(P_{n-2}, i-1) \neq \phi$ and $D_2(P_{n-3}, i-1) = \phi$ then
$D_2(P_n, i) = D_2(P_n, n-1) = \{[n] - \{x\} \,/\, x \in [n] \text{ and } x \neq 1, n\}$.
(iv) If $D_2(P_{n-1}, i-1) = \phi$, $D_2(P_{n-2}, i-1) \neq \phi$ then,
$D_2(P_n, i) = \{X \cup \{n\} / X \in D_2(P_{n-2}, i-1)\}$.
(v) If $D_2(P_{n-1}, i-1) \neq \phi$, $D_2(P_{n-2}, i-1) = \phi$ then,
$D_2(P_n, i) = \{Y \cup \{n\} \,/\, Y \in D_2(P_{n-1}, i-1)\}$.
(vi) $D_2(P_{n-1}, i-1) \neq \phi$ and $D_2(P_{n-2}, i-1) \neq \phi$ then, $D_2(P_n, i) = \{X \cup \{n\} \cup Y \cup \{n\}\}$
where $X \in D_2(P_{n-2}, i-1)$ and $Y \in D_2(P_{n-1}, i-1)$.

Proof :

(i) For every $n \geq 3$, $D_2(P_{2n-1}, n)$ has only one 2-dominating set as,
$D_2(P_{2n-1}, n) = \{1, 3, 5, 7, 9 \ldots\ldots\ldots\ldots 2n-1\}$.

(ii) Since $D_2(P_{n-2}, i-1) = \phi$, $D_2(P_{n-3}, i-1) = \phi$ and $D_2(P_{n-1}, i-1) \neq \phi$.
By Lemma 2.5, (ii) $i = n$. Therefore, $D_2(P_n, i) = D_2(P_{2n}, n) = [n]$.

(iii) Since, $D_2(P_{n-1}, i-1) \neq \phi$, $D_2(P_{n-2}, i-1) \neq \phi$ and $D_2(P_{n-3}, i-1) = \phi$.
By Lemma 2.5, $i = n-1$.
Therefore, $D_2(P_n, i) = D_2(P_n, n-1) = \{[n] - \{x\} \,/\, x \in [n] \text{ and } x \neq 1, n\}$.

(iv) Let X be a 2-dominating set of P_{n-2} with cardinality i – 1. All the elements of $D_2(P_{n-2}, n-1)$ end with n – 2. Therefore, n – 2 ∈ X, adjoin n with X. Hence, every X of $D_2(P_{n-2},i-1)$ belongs to $D_2(P_n,i)$ by adjoining n only.
Conversely, Suppose Z ∈ $D_2(P_n,i)$.
Here all the elements of $D_2(P_n,i)$ end with n only. Suppose n ∈ Z, then Z = X∪ $\{n\}$ where X ends with n – 2.

(v) Let Y be a 2-dominating set of P_{n-1} with cardinality i – 1. All the elements of $D_2(P_{n-1}, i-1)$ end with n – 1. Therefore, n – 1 ∈ Y, adjoin n with Y. Hence, every Y of $D_2(P_{n-1}, i-1)$ belongs to $D_2(P_n, i)$ by adjoining n only.
Conversely, Suppose Z ∈ $D_2(P_n,i)$.
Here all the elements of $D_2(P_n,i)$ end with n only. Suppose n ∈ Z, then Z = Y∪{n} where Y ends with n – 1.

(vi) Construction of $D_2(P_n, i)$ from $D_2(P_{n-1}, i-1)$ and $D_2(P_{n-2}, i-1)$.

Let X be a 2-dominating set of P_{n-2} with cardinality i – 1. All the elements of $D_2(P_{n-2}, i-1)$ end with n – 2. Therefore, n – 2 ∈ X, adjoin n with X. Hence, every X of $D_2(P_{n-2}, i-1)$ belongs to $D_2(P_n, i)$ by adjoining n only. Let Y be a 2-dominating set of P_{n-1} with cardinality i – 1. All the elements of $D_2(P_{n-1}, i-1)$ end with n–1. Therefore, n – 1∈ Y, adjoin n with Y. Hence, every Y of $D_2(P_{n-1}, i-1)$ belongs to $D_2(P_n, i)$ by adjoining n only.
Conversely, Suppose Z∈ $D_2(P_n, i)$.
Here all the elements of $D_2(P_n, i)$ end with n only.
Suppose n ∈ Z, then Z = {X ∪{n}⋃Y ∪{n}}where X ends with n – 2, X ∈ $D_2(P_{n-2}, i-1)$ and Y ends with n – 1, Y ∈ $D_2(P_{n-1}, i-1)$.

Theorem 2.7 :

If $D_2(P_n, i)$ be the family of the 2-dominating sets of P_n with cardinality i, where $i \geq \left\lceil \frac{n+1}{2} \right\rceil$ then, $d_2(P_n, i) = d_2(P_{n-1},n-1) + d_2(P_{n-2},n-1)$

Proof :

From Theorem 2.6, we consider all the four cases as given below, where $i \geq \left\lceil \frac{n+1}{2} \right\rceil$:

(i) If $D_2(P_{n-1}, i-1) = \phi$ and $D_2(P_{n-2}, i-1) = \phi$ then $D_2(P_n, i) = \phi$

(ii) If $D_2(P_{n-1}, i-1) = \phi$, $D_2(P_{n-2}, i-1) \neq \phi$ then, $D_2(P_n,i) = \{X\cup\{n\} / X \in D_2(P_{n-2}, i-1)\}$

(iii) If $D_2(P_{n-1}, i-1) \neq \phi$, $D_2(P_{n-2}, i-1) = \phi$ then, $D_2(P_n,i) = \{Y\cup\{n\} / Y \in D_2(P_{n-1}, i-1)\}$.

(iv) $D_2(P_{n-1}, i-1) \neq \phi$ and $D_2(P_{n-2}, i-1) \neq \phi$ then, $D_2(P_n, i) = \{X \cup \{n\} \cup Y \cup \{n\}\}$
where $X \in D_2(P_{n-2}, i-1)$ and $Y \in D_2(P_{n-1}, i-1)$

From the above construction in each case, we obtain that

$$d_2(P_n, i) = d_2(P_{n-1}, i-1) + d_2(P_{n-2}, i-1)$$

3. 2-DOMINATION POLYNOMIALS OF PATHS

Definition 3.1

Let $D_2(P_n, i)$ be the family of the 2-dominating sets of P_n with cardinality i and let $d_2(P_n,i) = |D_2(P_n, i)|$. Then, the 2-domination polynomial $D_2(P_n,x)$ of P_n is defined as, $D_2(P_n, x) = \sum_{i=\gamma_{2(P_n)}}^{n} d_2(P_n, i)x^i$, where $\gamma_2(P_n)$ is the 2-domination number of P_n.

Lemma 3.2

Let P_n, $n \geq 3$ be the path with $|V(P_n)| = n$ then,
(i) $D_2(P_n, i) = \phi$ if $i < \gamma_2(P_n)$ or $i > n$.
(ii) $D_2(P_n, x)$ has no constant term and first degree terms.
(iii) $D_2(P_n, x)$ is a strictly increasing function on $[0, \infty)$.

Proof :

(i) Since, $D_2(P_n, i) = \phi$, if $i < \gamma_2(P_n)$ and $D_2(P_n, n+k) = \phi$, $k = 1,2,3...$,
we have $d_2(P_n, i) = 0$ if $i < \gamma_2(P_n)$ or $i > n$.
Therefore, $D_2(P_n, i) = \phi$, if $i < \gamma_2(P_n)$ or $i > n$.

(ii) A single vertex cannot 2-dominate itself. So, the set of all vertices of P_n is 2-dominated by at least two of the vertices of P_n. Hence, the 2-domination polynomial has no constant term as well as first degree terms.

(iii) From the definition of 2-domination polynomial $D_2(P_n, x)$ is a strictly increasing function on $[0,\infty)$.

Theorem 3.3

For every $n \geq 5$

$D_2(P_n, x) = x\,[D_2(P_{n-1}, x) + D_2(P_{n-2}, x)]$ with initial values

$D_2(P_2, x) = x^2$

$D_2(P_3, x) = x^2 + x^3$

Proof :

We have $d_2(P_n, i) = d_2(P_{n-1}, i-1) + d_2(P_{n-2}, i-1)$

Therefore, $d_2(P_n, i)x^i = d_2(P_{n-1}, i-1)\,x^i + d_2(P_{n-2}, i-1)\,x^i$

$\sum d_2(P_n, i)\,x^i = \sum d_2(P_{n-1}, i-1)\,x^i + \sum d_2(P_{n-2}, i-1)\,x^i$

$\sum d_2(P_n, i)\,x^i = x \sum d_2(P_{n-1}, i-1)\,x^{i-1} + x \sum d_2(P_{n-2}, i-1)\,x^{i-1}$

$D_2(P_n, x) = x\,D_2(P_{n-1}, x) + x D_2(P_{n-2}, x)$

$D_2(P_n, x) = x\,[D_2(P_{n-1}, x) + D_2(P_{n-2}, x)]$

with the initial values

$D_2(P_2, x) = x^2$

$D_2(P_3, x) = x^2 + x^3$

We obtain $d_2(P_n, i)$ for $2 \leq n \leq 15$ and $2 \leq i \leq 15$ as shown in Table 1

Table 1

n/i	2	3	4	5	6	7	8	9	10	11	12	13	14	15
P_2	1													
P_3	1	1												
P_4	0	2	1											
P_5	0	1	3	1										
P_6	0	0	3	4	1									
P_7	0	0	1	6	5	1								
P_8	0	0	0	4	10	6	1							
P_9	0	0	0	1	10	15	7	1						
P_{10}	0	0	0	0	5	20	21	8	1					
P_{11}	0	0	0	0	1	15	35	28	9	1				
P_{12}	0	0	0	0	0	6	35	56	36	10	1			
P_{13}	0	0	0	0	0	1	21	70	84	45	11	1		
P_{14}	0	0	0	0	0	0	7	56	126	120	55	12	1	
P_{15}	0	0	0	0	0	0	1	28	126	210	165	66	13	1

Theorem 3.4 :

The following properties hold for the coefficients of $D_2(P_n, x)$.

(i) $d_2(P_n, n) = 1$, for every $n \geq 2$.
(ii) $d_2(P_n, n-1) = n-2$, for every $n \geq 3$.
(iii) $d_2(P_n, n-2) = \frac{1}{2}[n^2-7n + 12]$, for every $n \geq 5$.
(iv) $d_2(P_n, n-3) = \frac{1}{6}[n^3-15n^2 + 74n-120]$, for every $n \geq 7$.
(v) $d_2(P_n, n-4) = \frac{1}{24}[n^4-26n^3 +251n^2-1066n+1680]$, for every $n \geq 8$.
(vi) $d_2(P_{2n+1}, n+1) = 1$, for every $n \geq 1$.
(vii) $d_2(P_{2n}, n+1) = n$, for every $n \geq 2$.

Proof :

(i) Since, $D_2(P_n, n) = [n]$, we have the result.
(ii) Since, $D_2(P_n,n-1) = \{[n-1]-\{x\}/x\in [n] \text{ and } x \neq 1, n\}$,
we have $d_2(P_n, n-1) = n-2$.
(iii) To prove, $d_2(P_n, n-2) = \frac{1}{2}[n^2-7n + 12]$, for every $n \geq 4$.
we apply induction on n. when n=5,

L.H.S = $d_2(P_5, 5-2) = d_2(P_5, 3) = 1$ (from the table) and
R.H.S $= \frac{1}{2}[n^2 - 7n + 12]$

$= \frac{1}{2}[5^2 - 7 \times 5 + 12] = 1.$

Therefore the results is true for n=5. Now, suppose that the result is true for all numbers less than ‘n’ and we prove it for n.

By Theorem 3.3, we have

$d_2(P_n, n-2)$ $= d_2(P_{n-1}, n-3) + d_2(P_{n-2}, n-3)$

$= \frac{1}{2}[(n-1)^2 - 7(n-1) + 12] + n - 4$

$= \frac{1}{2}[n^2 - 2n + 1 - 7n + 7 + 12] + n - 4$

$= \frac{1}{2}[n^2 - 7n + 12]$

Hence the result is true for all n.

(iv) To Prove, $d_2(P_n, n-3) = \frac{1}{6}[n^3 - 15n^2 + 74n - 120]$, for every $n \geq 7$,

we apply induction on n. When n = 7,

L.H.S = $d_2(P_7, 7-3) = d_2(P_7, 4) = 1$ (from the table) and

R.H.S $= \frac{1}{6}[n^3 - 15n^2 + 74n - 120]$

$= \frac{1}{6}[7^3 - 15 \times 49 + 74 \times 7 - 120] = 1.$

Therefore the results is true for n = 7. Now, suppose that the result is true for all numbers less than 'n' and we prove it for n.

By Theorem 3.3, we have

$d_2(P_n, n-3) = d_2(P_{n-1}, n-4) + d_2(P_{n-2}, n-4)$

$= \frac{1}{6}[(n-1)^3 - 15(n-1)^2 + 74(n-1) - 120] + \frac{1}{2}[(n-2)^2 - 7(n-2) + 12]$

$= \frac{1}{6}[n^3 - 3n^2\ 3n - 1 - 15n^2 + 30n - 15 + 74n - 194 + 33n^2 - 12n + 12 - 21n + 42 + 36]$

$= \frac{1}{6}[n^3 - 15n^2 + 74n - 120].$

(v) To Prove, $d_2(P_n, n-4) = \frac{1}{24}[n^4 - 26n^3 + 251n^2 - 1066n + 1680].$

For every $n \geq 9$, we apply induction on n.

when n=9,

L.H.S $= d_2(P_9, 9-4) = d_2(P_9, 5) = 1$ (from the table) and

R.H.S $= \frac{1}{24}[9^4 - 26 \times 9^3 + 251 \times 9^2 - 1066 \times 9 + 1680]$

$= \frac{1}{24}[6561 - 18954 + 20331 - 9594 + 1680]$

$= \frac{1}{24}[24]$

$= 1$

Therefore the results is true for n = 9.

Now, suppose that the result is true for all numbers less than 'n' and we prove it for n.

By Theorem 3.3, we have

$d_2(P_n, n-3) = d_2(P_{n-1}, n-5) + d_2(P_{n-2}, n-5)$

$= \frac{1}{24}[(n-1)^4 - 26(n-1)^3 + 251(n-1)^2 - 1066(n-1) + 1680] + \frac{1}{6}[(n-2)^3 - 15(n-2)^2 + 74(n-2) - 120]$

$= \frac{1}{24}[n^4 - 4n^3 + 6n^2 - 4n + 1 - 26(n^3 - 3n^2 + 3n - 1) + 251(n^2 - 2n + 1) - 1066n + 1680] + \frac{1}{6}[n^3 - 6n^2 + 24n - 8 - 15(n^2 - 4n + 4) + 74n - 148 - 120]$

$= \frac{1}{24}[n^4 - 26n^3 + 251n^2 - 1066n + 1680 - 4n^3 + 84n^2 - 584n + 1344 + 4n^3 - 84n^2 + 584n - 1344]$

$= \frac{1}{24}[n^4 - 26n^3 + 251n^2 - 1066n + 1680]$

(vi) Since, $D_2(P_{2n+1}, n+1) = \{1, 3, 5, 7, 9 \ldots\ldots\ldots (2n+1)\}$

we have, $d_2(P_{2n+1}, n+1) = 1$

(vii) To prove, $d_2(P_{2n}, n+1) = n$, for every $n \geq 1$,

We apply induction on n.
When n = 1,
L.H.S. = $d_2(P_2, 2) = 1$ (from the table) and
R.H.S. = n = 1.
Therefore, the result is true for n = 1

Now, suppose that the result is true for all numbers less than, n + 1 and we prove it for n + 1.

By Theorem 3.3, We have, $D_2(P_{2n}, n+1) = d_2(P_{2n-1}, n) + d_2(P_{2n-2}, n)$

$= 1 + n - 1$

$= n$

Therefore, the result is true for all $n \geq 3$

Conclusion :

In this paper, 2-domination polynomials of paths has been derived by identifying its 2-dominating sets. It also help us to characterize the 2-dominating sets of cardinality i. We can generalize this study to any power of path and some interesting properties can be obtained via the roots of the 2-domination polynomial of P_n.

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