Study of the Inverse Problem in the Theory of Eigen Function Expansion Associated with Second-Order Differential Equations

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Abstract:*In this present paper, we studied about the inverse problem (i.e. the construction of the differential equation from the given spectral function) in the theory of eigenfunction expansion associated with second-order differential equations occurs in the works of Ambarzuian as far boack as 1929. Subsequently it was considered among others by Marcenko, Krein, Gelfand and Levitan, Blokh, Feddev, Levinson, Hochstadt, Butler, Bellman and Richardson and Gasymov and Levitan. As far as the knowledge of the present author goes, the inverse problem has been dealt with mostly by Russian mathematicians. A short account of the theory developed so far by some authors is given here.*

Keywords: **Inverse problem, Differential Equation, Eigenfunction, Spectral Theorem.**

1. Introduction

The modern theory of singular differential operator was first developed by N.Neyl (1885-1955) on singular self-and joint Linear differential operator of the second order and later on developed by M.H. Stone, J.VonNewmann (1905-1957), K. Friedrichs, K.Kodaira.Hilbert took up the discussions on a pair of simultaneous differential equations of the second order and Whyburn, Kamke,. Lidskii, Levin, Kodaira, Coddinton and Levinson, Chakravarty, Bhagat, Tiwari studied problems with two (or more) simultaneous second order differential equations and advanced the theory to a large extent in what follows we sketch in brief only a few previous works on zeros of eigenfunctions and related works. Titchmarsh in 1944 discussed the finite case of the simultaneous system of two first-order linear differential equations and he considered the extension to the infinite case in 1941. Context Sanger discussed two first-order equations in 1953 and 1954.

2. Inverse Problem

Levinson [1949] proves in one of the theorems that there can be at most one $P(x) \in L(0, \pi)$ such that the equation

$$y'' + (\lambda - p(x))\, y = 0 \qquad (1)$$

has two assigned set of characteristic values (eigenvalues) for the two pair of boundary conditions

(1) $y(0)\cos\alpha + y'(0)\sin\alpha = y(\pi)\cos\beta + y'(\pi)\sin\beta = 0$

(2) $y(0)\cos\alpha + y'(0)\sin\alpha = y(\pi)\cos\gamma + y'(\pi)\sin\gamma = 0$

provided that $\sin(\beta-\gamma) \neq 0$.

Gelfand and Levitan [1951] dealt with the inverse problem for the system (I) viz.

$$y'' + (\lambda - q(x)\, y = 0 \qquad (2)$$

with boundary conditions

$$y(0) = 1,\ y'(0) = h \qquad \text{(Ia)}$$

The function q(x) is assumed to be continuous on any finite interval. They make use of the result viz., "There exists a monotonic function $\rho(\lambda)$, bounded on each finite interval such that for any function $f(x) \in L^2(0,\infty)$

$$\int_0^{\infty} f(x)dx = \int_{-\infty}^{\infty} E^2(\lambda)\, d\rho(\lambda)$$

$$E(\lambda) = \int_0^\infty f(x)\phi(x,\lambda)\,dx$$

$\phi(x,\lambda)$ being the solution of (I) with the initial conditions (Ia)". The function $\rho(\lambda)$ is the "Spectral function" of the system (I) with the condition (Ia). A method of actual computation of q(x) has also been given. The problem is attached with elementary means by reducing it to the solution of a certain integral equation. Form the spectral function $\rho(\lambda)$ theeigenfunction$\phi(x,\lambda)$ is constructed by orthogonalising the function $\cos\sqrt{\lambda}t$ with respect to $\rho(\lambda)$ in a way similar to that is which polynomials are constructed by orthogonalising the powers of x.

Blokh [1953] takes up the operator

$l(y) = - y'' + q(x)\, y,\ 0 \leq x < \infty.$

with the boundary condition

$y'(0) - \theta\, y(0) = 0$

and obtains solutions of the corresponding inverse problem over the interval $(-\infty,\infty)$.

Faddev [(59), 1959] deals with the inverse problem in the quantum theory of scattering for the system.

$- y'' + q(x)\, y = \lambda y,$ (3)

$v(L,s) = 0\ \partial v/\partial x = z(x)\, vx \mid_0$

z(s) being a meromorphic function of a.

Levitan [1963] deals with the inverse problem associated with the system (I) viz.,

$y'' + (\lambda - q(x))\, y = 0.$

with the boundary conditions $y'(0) - h\, y(0) = 0$, $y'(\pi) + h\, y(\pi) = 0$

which give one set of eigenvalue and a second system which consists of the same differential equation, the same boundary condition as x = 0 but a different boundary condition at $x = \pi$i.e $y'(\pi) + H\, y(\pi) = 0$, giving another system of eigenvalues. His approach is different from that of M.G.Krein who solved the same problem as far back as 1951.

Gasymov and Leviten [1964] consider the inverse problem for the system (I) viz.,

$-y'' + q(x)\, y = \lambda y$

with the boundary $y'(0) - h\, y(0)$. Their method is the use of the theory of linear integral equations and is different from that of Galfand and Levitan.

Hochstadt [1967] uses a method of Levinson to established that "givento Sturm- Liouville problems.

(1) $y'' + (\lambda - q(x))\, y = 0$

(2) $y(0)\cos\alpha + y'(0)\sin\alpha = y(\pi)\cos\beta + y'(\pi)\sin\beta = 0$

(2) $y(0)\cos\alpha + y'(0)\sin\alpha = y(\pi)\cos\gamma + y'(\pi)\sin\gamma = 0$

where q(x) is real and integrable in $[0,\pi]$, $\sin(\gamma-\beta) \neq 0$, them the two spectra corresponding to the two problems uniquely determines q(x), almost everywhere".

Hochatadt further proves that the spectral function of the boundary value problem.

(4) $y'' + (\lambda - q(x))\, y = 0$

(5) $y(0) + ay'(\pi) + \lambda y(\pi) = 0$

where a is real but not equal to zero, and q(x) is integrable in $[0,\pi]$, determines uniquely q(x) almost everywhere.

Brodakii [1957] is concerned with the inverse problem associated with the system.

$\frac{dy_i}{dx} = \frac{i}{\lambda} b_{ij}(x) y_j \quad (1 \leq I \leq n,\ \ 0 < x \leq l)$ (4)

Gasymov and Levitan [1966] discuss the inverse problem for the de Dirao system of differential equations.

$\{B\, d/dx + q(x)\}\, y = \lambda y, \quad (0 \leq x < \infty)$ (5)

where

$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, q(x) = B = \begin{pmatrix} p(x) & q(x) \\ q(x) & r(x) \end{pmatrix},\ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

and p(x), q(x), r(x) are real functions integrable over $[0,\infty]$.

Butler [1968] takes up an inverse problem for differential operators of fourth order with rational co-efficient.

Other workers in the line are Sadovincii [1972], Baranova [1972] and Sadovincii [1973].

A short theory of the inverse problem involving second-order linear differential equations and a system of first order Dirac-type equations also occur in Nainark, where also a good number of references on the problem can be found.

In the present paper as already stated, the operator considered is similar to that of Chakravarty and in the same as that considered by Tiwari (1964).

3. Eigen Function Expansions

If f (x) is continuous, then

$$f_1(x) = -\frac{1}{\pi i}\lim_{R \to \infty}\int_{-R+i\epsilon}^{R+i\epsilon} \Phi_1(x,\lambda)\, d\lambda \qquad (6)$$

Similarly

$$f_2(x) = -\frac{1}{\pi i}\lim_{R \to \infty}\int_{-R+i\epsilon}^{R+i\epsilon} \Phi_2(x,\lambda)\, d\lambda \qquad (7)$$

The above regulation is true uniformly for $0 < \epsilon \leq 1$

In this paper we also investigate the behavior of the integrals (6) and (7) and (7) as $\epsilon \to 0$.

Here we discuses (6) and the same arguments will apply to (7). First of all we show that (6) can be replaced by

$$f_1(x) = -\lim_{R \to \infty}\left[\frac{1}{\pi}\int_{-R-i\epsilon}^{R+i\epsilon} \mathrm{im}\ \Phi_1(x,\lambda) d\lambda\right] \qquad (8)$$

Since $\Phi_1(x,\lambda)$ is analytic in the upper and lower half planes, it follows from the convergence theorem that

$$f_1(x) = -\frac{1}{\pi i}\lim_{R \to \infty}\int_{-R-i\epsilon}^{R+i\epsilon} \Phi_1(x,\lambda) d\lambda \qquad (9)$$

Let $\lambda = s - i\epsilon = \bar{\lambda}$-2 $i\epsilon$, ϵ being fixed. Then from (4.8.2), we have

$$f_1(x) = -\frac{1}{\pi i}\lim_{R \to \infty}\int_{-R+i\epsilon}^{R+i\epsilon} \Phi_1(x, \bar{\lambda} - 2i\,\epsilon) d\bar{\lambda}$$

$$= \frac{1}{\pi i}\lim_{R \to \infty}\int_{-R+i\epsilon}^{R+i\epsilon} \Phi_1(x, \bar{\lambda}) d\bar{\lambda} \qquad (10)$$

Adding (6) and (10) we have

$$2f_1(x) = -\frac{1}{\pi i}\lim_{R \to \infty}\int_{-R+i\epsilon}^{R+i\epsilon} \Phi_1(x,\lambda) - \Phi_1(x,\bar{\lambda})\ d\bar{\lambda}$$

$$= \frac{2}{\pi i}\lim_{R \to \infty}\int_{-R+i\epsilon}^{R+i\epsilon} [\mathrm{im}\ \Phi_1(x,\lambda)] d\lambda$$

becauseim $\Phi(x,\bar{\lambda}) = \Phi(x,\lambda)$. The proves (8)

4. Conclusion

The vector $\phi(x,\lambda)$ has been defined which satisfies the non-honogenous equation associated with the homogenous equation.The problems we discuss is the problem of determining the differential system when the spectral matrix is given (the inverse problem). In the present paper we dealing with the spectrum we have made use of Titchmarah's complex variable methods and we have dealt the inverse problem by following the methods of Gaaymov and Levitan and of Gelfand and Levitan for the second order differential systems.

References

[1]. *Bhagat, B.'Eigen function expansions associated with a pair of second order differential equations.'Proc. National Inst. Sciences of India. Vol. 35,A, No. 1 (1969)*

[2]. *Bhagat, B.'Some problems on a pair of singular second order differential equations' Ibid. 35A (1969), 232 -244.*

[3]. *Bhagat, B.'A spectral theorem for a pair of second order singular differential equations' Quart. J. Math. Oxford 21, (1970), 487-495.*

[4]. *Conte, S. D.andSangren, W.C.'On asymptotic solution for a pair of singular first order equations' Proc. Amer. Math. Soc. 4, (1953) 696-702*

[5]. *Pandey, Y.P.'Some problems on eigenfunction expansion associated with a pair of second order differential equations.' Thesis, Bhag. Univ. Bihar*

[6]. *Pandey, Y. P.and Kumar, A. 'Spectral theorem and eigenfunction expansion associated with a pair of second order differential equations.' Thesis, T. M. Bhag. Univ. , Bihar*

[7]. *Rose, B. W.andSangren, W. C.'Expansions associated with a pair of a singular first order differential equations' J. Math. Physics 4 (1963) 999-1008.*

[8]. *Titchmarsh, E.C.'Eigenfunction expansions associated with second order differential equation' Part I, Oxford 1962*

[9]. *Khoff, G.D. and Langer, R.E. ' The boundary value problem and developments associated with asystem of ordinary differential systems of the first order'. Proc. Amer. Acad. of Arts and Sciences, 58(1923): 51-128.*

[10]. *Bhagat, B., 'Eigenfunction expansions associated with a pair of second order differential equations'.Proc. National Inst. Sciences of India Vol. 35, A.No.1 (1969).*

[11]. *Bhagat, B., 'Some problems on a pair of singular second order differential equations'. Ibid 35A (1969): 232-44.*

[12]. *Bhagat, B., 'Ph.D. Thesis'. Patna University. 1966.*

[13]. *Chakravarty, N.K. 'Some problems in eigenfuction expansions I' Quart. J. Math. (16) 62 (1965) 135-50.*

[14]. *Conte, S.D. and Sangern. W.C., ' An asymptotic solution for a pair of singular first order equations' Proc. Amer. Math. Boc. 4(1953): 696-702.*

[15]. *Conte, S.D. and Sangern. W.C., 'An expansion theorem for a pair of first order equation' Canadian J. Math. 6 (1954): 554-60*

[16]. *Choudhary, J. and Everitt. W.ll. 'The spectrum of a fourth order differential operation' Proc. Royal Soc. Of Edinburgh Sec. Vol. LXVIII Part III No. 13 (193-210).*