Generalized T_F −Kannan fixed point theorem on metric spaces

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Abstract: The purpose of this paper is to establish the generalization of T_F −Kannan contractive mapping theorem on complete metric spaces.Our results extend and generalize the common fixed point result of M.Kir& H. Kiziltunc [10].

Key words: Fixed point, Common fixed point,$T_F -$ Kannan contractive mapping,Sequentially convergent, metric space.

1. Introduction:

In 1922, Banach, S. [1] proved the following his famous theorem, which ensures the existence and uniqueness of the fixed point.

Theorem 1.1: Let (X, d) be a complete metric space with itself maps $T: X \to X$. Then the maps T is called contraction if there exists $k \in [0,1)$ such that

$d(Tx, Ty) \leq kd(x, y)$(1)

For all $x, y \in X$. If the metric space (X, d) is complete, then the mapping satisfying (1) has a unique fixed point. It is the first important result on fixed points for contractive mappings. The theorem 1.1, is known as Banach contraction mapping theorem or Banach fixed point theorem and it is a forceful tool in nonlinear analysis. There have been a lot of fixed point results dealing with mappings satisfying various types of contractive inequalities. These generalizations were made either by wakening the contractive condition or by imposing some additional condition on ambient space. So, it is clear that, the inequality (1) implies the continuity of T.

In 1968, Kannan [2] establish a fixed point theorem as follows:

Theorem 1.2: If a mapping $T: X \to X$, where (X, d) is a complete metric spaces, satisfies the inequality

$$d(Tx, Ty) \leq k((d(x, Tx) + d(y, Ty)) \quad (2)$$

where $k \in [0, {}^{1}/_{2})$ and for all $x, y \in X$, then T has a unique fixed point. The mapping satisfying (2) are called Kannan type mappings. There is a large literature dealing with Kannan type mappings. A similar contractive condition has been introduced by Chatterjea [3] in 1972.

Rhoades,B. E. [4] proved fixed point theoremsmade by a comparison of various type of contraction mappings. In 2009, Beiranvand,A. et al.[5]introduced new classes of contractive functions as T- contraction and T- contractive mappingsand established the Banach contraction principle. In sequel, Moradi, S. [6] introduced the T-Kannan contractive mapping, which extended the well know fixed point theorem, which is givenin [2]. Dutta and Choudhary [7] also generalized contraction principle in metric spaces.

In 2010, Moradi and Beiranvand [8] also introduced TF - contraction mappings as follows:

Definition 1.3: Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be sequentially convergent, if we have for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence, then $\{y_n\}$also is convergent.$T$is subsequentially convergent if we have, every sequence $\{y_n\}$,if $\{Ty_n\}$is convergence then $\{y_n\}$has aconvergent subsequence.

Definition 1.4: Let (X, d) be a metric space and $f, T : X \to X$ be any two mappings .Then f issaid to be a T_F -contraction , if there exists $k \in [0,1)$ such that for al $x, y \in X$

$$F\big(d(Tx, Ty)\big) \leq kF\big(d(x, y)\big), \qquad (3)$$

where

(1) $F: [0, \infty] \to [0, \infty)$, F is nondecreasing continuous from the right and $F^{-1}(0) = \{0\}$.

(2) T is one to one, continuous and subsequentially convergent.

If f is a $T_F -$ contraction mappingthen by condition(3), f has a unique fixed point in complete metric space (X, d).

Moradi and Davood [9] proved a new extension of Kannan fixed point theorem on complete metric and generalized metric space, which is extend the well-known results of Moradi, S. [6].

Recently, in 2014, Mehmet, Kir and H. Kiziltunc [10] proved unique fixed point results for the notion of T_F-contractive conditions are investigated for Kannan and Chatterjea type mappings.

In this paper, we study and discuss some unique common fixed point theorem for generalized T_F- Kannan type contractive mappings in metric space. Our main results greatly generalize the previous work in the literature of [10] of theorem 2.1.

2. Main Result

Theorem 2.1. [Generalized T_F– Kannan contractive mapping theorem): Let (X, d) be a complete metric space $T, R, S: X \to X$ be mapping such that T is one to one, continuous and subsequently convergent. If $\lambda \in [0, 1/2)$ and

$F(d(TRx, TSy)) \leq \lambda\,(F(d(Tx, TRx) + F(D(Ty, TSy))$ (2.1)

For all $x, y \in X$ and where $F: [0, \infty) \to [0, \infty)$ is nondecreasing continuous from the right and $F^{-1}(0) = \{0\}$. Then Rand S have a unique common fixed point. Also, if T is sequentially convergent. Then for every $x_0 \in X$ the sequence of iterates $\{R^{2n}x_0\}$ and $\{s^{2n+1}x_0\}$ converges to the common fixed point.

Proof: Let x_0 be an arbitrary point in X.We construct the iterative sequence $\{x_{2n}\}$ and$\{x_{2n+1}\}$ by

$x_{2n} = Rx_{2n-1} = R^{2n}x_0$, for $n = 01,2 \ldots ....$

and

$x_{2n+1} = Sx_{2n} = s^{2n+1}x_{0,}$ for,$n = 0,1,2,3 \ldots \ldots ..$

Using the inequality (2.1), we have

$$F\big(d(Tx_{2n,}\, Tx_{2n+1})\big) \leq F(d(TRx_{2n-1}, TSx_{2n}))$$
$$\leq \lambda\big[F\big(d\big(Tx_{2n-1,}\, TRx_{2n-1}\big)\big) + F\big(d(Tx_{2n}, TSx_{2n})\big)\big]$$
$$\leq \lambda\big[F\big(d\big(Tx_{2n-1,}\, Tx_{2n}\big)\big) + F\big(d(Tx_{2n}, Tx_{2n+1})\big)\big] \quad (2.2)$$

Similarly,

$$F\big(d(Tx_{2n+1,}\, Tx_{2n+2})\big) \leq F(d(TRx_{2n}, TSx_{2n+1}))$$
$$\leq \lambda\big[F\big(d\big(Tx_{2n,}\, TRx_{2n}\big)\big) + F\big(d(Tx_{2n+1}, TSx_{2n+1})\big)\big]$$
$$\leq \lambda\big[F\big(d\big(Tx_{2n,}\, Tx_{2n+1}\big)\big) + F\big(d(Tx_{2n+1}, Tx_{2n+2})\big)\big] \quad (2.3)$$

Therefore, we have

$$F\big(d(Tx_{2n,}\, Tx_{2n+1})\big) \leq \frac{\lambda}{1-\lambda} F\left(d\big(Tx_{2n-1,}\, Tx_{2n}\big)\right) \quad (2.4)$$

and

$$F\big(d(Tx_{2n+1,}\, Tx_{2n+2})\big) \leq \frac{\lambda'}{1-\lambda'} F\left(d\big(Tx_{2n,}\, Tx_{2n+1}\big)\right) \quad (2.5)$$

We can conclude, be repeating the same argument that,

$$F\big(d(Tx_{2n,}\, Tx_{2n+1})\big) \leq \frac{\lambda}{1-\lambda} F\left(d\big(Tx_{2n-1,}\, Tx_{2n}\big)\right)$$
$$\leq \left(\frac{\lambda}{1-\lambda}\right)^2 F\left(d\big(Tx_{2n,}\, Tx_{2n-1}\big)\right)$$
$$\leq \left(\frac{\lambda}{1-\lambda}\right)^3 F\left(d\big(Tx_{2n-1,}\, Tx_{2n-2}\big)\right)$$

$\leq \ldots\ldots\ldots \leq$

$\leq\left(\frac{\lambda}{1-\lambda}\right)^{2n} F\left(d\left(Tx_{0,} Tx_1\right)\right)$

Put $\frac{\lambda}{1-\lambda} = h$.Then we have

$F\left(d(Tx_{2n,} Tx_{2n+1})\right) \leq h^{2n} F\left(d\left(Tx_{0,} Tx_1\right)\right)$ (2.6)

Similarly, wehave

$$F\left(d(Tx_{2n+1,} Tx_{2n+2})\right) \leq \left(\frac{\lambda'}{1-\lambda'}\right)^{2n+1} F\left(d\left(Tx_{0,} Tx_1\right)\right)$$

Put $\frac{\lambda'}{1-\lambda'} = h'$. Then we have

$F\left(d(Tx_{2n+1,} Tx_{2n+2})\right) \leq (h')^{2n+1} F\left(d\left(Tx_{0,} Tx_1\right)\right)$ (2.7)

By (2.6), for every $m, n \in N$ such that $m > n$, we have

$F\left(d(Tx_{2n,} Tx_{2m})\right) = \left[F\left(d\left(Tx_{2n,} Tx_{2n+1}\right) + d(Tx_{2n+1}, Tx_{2n+2}) + \cdots .. + d(Tx_{m-1}, x_{2m})\right)\right]$

$\leq \quad F(h^{2n} + h^{2n-1} + \cdots \ldots \ldots \ldots \ldots \ldots + h^{2m-1}) d\left(Tx_{0,} Tx_1\right)$

$\leq \left(\frac{h}{1-h}\right)^{2n} F\left(d\left(Tx_{0,} Tx_1\right)\right)$ (2.8)

Letting $m, n \to \infty$in (2.8), we have $F\left(d(Tx_{2n,} Tx_{2m})\right) \to \infty$as $m, n \to \infty$. So, $d(Tx_{2n,} Tx_{2m}) \to \infty$ as $m, n \to \infty$. Thus, $\{Tx_{2n}\}$ is a Cauchy sequence in metric space (X, d). Since X is a complete metric space, there exists $v \in X$ such that

$\lim_{n\to\infty} Tx_{2n} = v$ (2.9)

Since T is subsequentially convergent and $\{x_{2n}\}$ has a convergent sub sequence. So, there exists

$u \in X$and $\{x_{2ni}\}$such that

$\lim_{i\to\infty} x_{2ni} = u$ (2.10)

Since T is continuous. Then from (2.10), we have

$\lim_{i\to\infty} Tx_{2ni} = Tu$ (2.11)

Now, from (2.9) and (2.11), we get

$Tu = v$ (2.12)

So, consider

$F(d(Tu, TRu)) \leq F(d(Tu, Tx_{2ni}) + d(Tx_{2ni}, TRu))$

$= F\left(d(Tu, Tx_{2ni}) + d(TR^{2ni}x_0, TRu)\right)$

$\leq F\left(d(Tu, Tx_{2ni}) + d(TR^{2ni}x_0, TR^{2ni}x_1)\right) + d(TR^{2ni}x_1, TRu))$

$= F(d(Tu, Tx_{2ni}) + d(TRx_{2ni}, TRx_{2ni+1})) + d(TR^{2ni}x_1, TRu)) \to 0$

$(i \to \infty)$. Therefore, $F(d(Tu, TRu)) = 0 \Rightarrow d(Tu, TRu) = 0. \Rightarrow Tu = TRu$. Also, T is one to one. Therefore, $Ru = u$. Thus, u has a unique fixed point of R.

Now if Tis sequentially convergent by replacing $\{2n\}$with $\{2n_i\}$.We conclude that $\lim_{n\to\infty} x_{2n} = u$and this shows that $\{x_{2n}\}$converges to the fixed point of R. Similarly, it can be established that $\{x_{2n+1}\}$converges to the fixed point of S. $i.e. \lim_{n\to\infty} x_{2n} = u = \lim_{n\to\infty} x_{2n+1.}$

This complete the proof of the theorem.

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