Study and some Result on Non expansive Mapping in linear 2 normed spaces.

Dr. Hans Kumar Singh

Department of Physics
Baboo Bhuneshwar Prasad Degree College, Jai Prakash University, Chapra.
hksingh.hiht@gmail.com

INTRODUCTION— The notion of linear 2- normed spaces was introduced by S. Gahler. He further studies the topological studies of 2-normed spaces. Iseki introduced the notion of non-expansive mapping in 2- normed spaces. Then mathematician like Diminni and white further studied non-expansive mapping in linear 2- normed spaces and obtained the results of Iseki as their corollaries and they contributed a lot for the extension of this branch of mathematics, physics and other Science.

KEYWORD — 2- normed spaces, non-expensive mapping, convex subset

1. Let X be a linear space of dimension greater than 1 and let $\|\cdot,\cdot\|$ be a real valued function defined on X × X such that :
   1. $\| a, b\| = 0$ if any only if and b are linearly dependent,
   2. $\| a, b\| = \|b, a\|$,
   3. $\|a, \alpha b\| = | \alpha | \|a, b\|$, were α is real,
   4. $\|a + b, c\| < \|a, b\| + \|a, c\|$.

   $\|\cdot,\cdot\|$ is called a 2-norm on X and (X, $\|\cdot,\cdot\|$ is a linear 2-normed space. By condition 2 and 4, a 2-norm is non-negative.

Definition **:** If K is a convex subset of X, a mappings $T : K \to X$ is said to be non-expansive if for every $x, y \in K$ and $z \in X$,

1. $\|T(x) - T(y), z\| \leq \|x - y, z\|$.

   In the following, the real number system will be denoted by R. Also, a subset of L of x of the form $\{x_1 + ax_2 : a \in R\}$, where x_2 is non-zero, will be called a line. $\alpha \in$

Theorem **:** Let K be a convex set which contains a least 2 elements and is none a subset of line. Then, T is non-expansive if and only if there is a $c \in R$ and there is a point $z_0 \in X$ such that $|c| < 1$ and $T(x) = cx + z_0$, for every $x \in K$.

Proof- Since all functions of the above type are non-expansive, we need show only that all non-expansive maps are of this type.

1. Assume first the $0 \in K$ and T (o) = 0. Then, for every $x \in X$,
2. ||T (x), z||<|| X, Z||.

Therefore, for each $x \in K$, there is a real number g(x) such that T(x) = g(x)x.

If x and y are independent elements of K, then $\frac{1}{2}$ (x + y) $\in$ K also, and by (1),

$$\left(T\frac{x+y}{2}\right)--T(x), x--y\,|||<||\,\frac{x+y}{2}x-y\,||=0.$$

Therefore, there is a $k \in R$ such that

$$\left(T\frac{x+y}{2}\right)--T(x)=k(x-y)$$

$$g\left(\frac{x+y}{2}\right)\left(\frac{x+y}{2}\right)-g(x)x=k(x-y).$$

Then,

$$\left[\frac{1}{2}g\left(\frac{x+y}{2}\right)-g(x)-k\right]x=-\left[k+\frac{1}{2}g\left(\frac{x+y}{2}\right)\right]y$$

which implies that g(x) $=g\left(\frac{x+y}{2}\right)$ by the independence of x and y. Since a similar argument shows that g(y)= $g\left(\frac{x+y}{2}\right)$, it follows g(x) = g(y) whenever x and y are independent.

If x and y are non-zero, independent elements of K, then since K is not a subset of a line, there is a $z \in K$ such that z and x and z and y are independent. By the arguments used above, g(x) = g(z) = g(y).

Therefore, g(x) = g(y) for all non-zero x, $y \in K$. Since T(0) = 0, there is a real number c such that T (x) cx for every $x \in K$. Finally, (2) implies that |c| < 1.

2. For arbitrary T and K which satisfy the hypotheses, choose and $x \in K' = \{x - x_0 : x \in K\}$. Then K' is not contained in a since K is not a subset of a line, and $x \in K'$. Define $S : K' \to x$ by

$||S(x - x_0) - S(y - x_0), z|| = ||\, T(x) - T(y), z\,||$.

$< ||x - y, z||$

$= ||(x - x_0) - (y - x_0), z||$.

Hence, S is non-expansive on K' and

$S(0) = S(x - x_0) = T(x_0) - T(x_0) = 0$

By part 1, there is a $c \in R$ such that $|C| < 1$ and for every $x \in K$,

$S(x - x_0) = c(x - x_0)$.

Therefore, for every $x \in K$,

$T(x) = cx + T(x_0) - x_0$.

The following example shows that the characterization fails if K is contained in a line.

Example: Suppose K is subset of the line $L = T(x) = cx + T(x_0) - x_0$.

Define $T : K \to X$ by $T(x_1 + \alpha x_2) = (\sin\alpha)x_2$.

Then, if $x_1 + \alpha x_2$ and $x_1 + \gamma x_2$ are in K and $z \in X$,

$\| T(x_1 + \alpha x_2) - T \| (x + \gamma x_2), z \| = \| \sin\alpha - \sin\gamma \| \| x_2, z \| < \alpha - \gamma \| \| x_2, z \|$

$= \| (x_1 + \alpha x_2) - (x + \gamma x_2), z \|$.

Hence, T is a non-expansive mapping which does not satisfy Theorem 1.

For convex sets which are subsets of lines, we have the following characterization of non-expansive mappings.

***Theorem*:** Suppose K is a convex subset of line $L = \{x_1 + \alpha x_2 : \alpha \in R\}$, where $x_1 \in K$, and let $\{\alpha : x_1 + \alpha x_2 \in R\}$. Then, $T : K \to X$ is non-expansive if and only if there is a function $g : A$ $g(0) = 0$ and $T(x_1 + \alpha x_2) = g(\alpha) x_2 + T(x_1)$.

***Proof*:** Again, since the sufficiency of the above conditions is clear, we deed only to prove the necessity.

1. Assume $X_1 = 0$ and $T(0) = 0$. Then, for every $\alpha \in A$ and $z \in X$, (3) $\|T(x_2), z\| < \| \alpha x_2, z\|$.

   Therefore, for every non-zero $\alpha \in A$, there is a real number $g(\alpha)$ such that $|g(\alpha) - g(\gamma)| < |\alpha - \gamma|$ for every $\alpha, \gamma \in A$.

2. If $x_1 = 0$ or $T(x_1) = 0$ let $K' = \{\alpha x_2 : \alpha \in A\}$. Then, K' is convex, $0 \in K'$, and $K' = \{\alpha x_2 : \alpha \in R\}$. Define $S : K' \to X$ by

   $S(\alpha x_2) = T(x_1 + \alpha x_1) - T(x_1)$

   for every $\alpha \in A$. Note that $S(0) = 0$ and for $\alpha, \gamma \in A$ and $z \in X$,

   $\| S(\alpha x_2) - - S(\gamma x_2), z \| = \| T(x_1 + \alpha x_2) - - T(x_1 + \gamma x_2), z \| < \| \alpha x_2 - \gamma x_2, x \|$.

Therefore, since S and K' satisfy the assumptions made in part 1, it follows that there is a function g : A $\to$ R such $S(\alpha x_2) = g(\alpha)x_2$. Hence, for every $\alpha \in$ A, $T(x_1 + \alpha x_2) = g(\alpha)x_2 + T(x_1)$.

It is known that in a strictly convex 2-normed space, the set F(T) of fixed points of a non-expansive T is always a convex set. This result can now be proven for any 2-normed space.

REFERENCES-

1. J. Baker, Isometrics in Normed Spaces, Amer Math Monthly,(1971), 78, 655-657.
2. Y. J. Cho. et. al. Strictly 2-convex linear 2-normed spaces, Math Japonica, (1982), 26, 495-498.
3. Y. J. Cho. et. al. Strictly 2-convex linear 2-normed saces, Math Japonica,(1982), 27, 609-612.
4. J. A. Clarkson, Uniformly convex spaces, Trans. Amer Math Soc.(1936), 396-414.
5. C. Diminnie , An example of a linear isometry in 2-normed spaces. Science Studies, St. Bonaventure University. (1973),29,1-4.
6. C. Diminmie et. al., Math Nachr,(1974), 319-324.
7. C. Diminnie and A. White Non-expansive Mappings in Linear 2-normed Spaces, Math, Sem. Kobe University.(1976), 4
8. C. Diminnie and A. G. White, Non-expansive mappings in Linear 2-normed spaces, Math Japonica,(1976), 21, 197-200.
9. R.E. Ehret , Linear 2-normed spaces, Doctoral Thesis Dissertation, St. Louis University. (1969)
10. I. Franic, Two results in 2-normed spaces, Glas Mat.(1982), II, 17,271-275.
11. Das E Sarasa, on statically Pre cauchey sequence Taiwanese J. Math (2014),18.1
12. H Lahlic, S. Ersan, Strongly lagunary ward continuity in 2 normed spaces, sci world J.(2014), 479679
13. H. Lakollic, S. Ersan, New type of continuity in 2 normed spaces, filomat (2016), 5.30 : 3
14. S. H. Khan and N. Hussain, "Convergence theorems for nonself asymptotically non expansive mappings," Computers & Mathematics with Applications,(2008), 55, 11, 2544–2553,

15. C. Chidume, E.Ofoedu, and H. Zegeye, "Strong and weak convergence theorems for asymptotically non expansive mappings," Journal of Mathematical Analysis and Applications,(2003), 280, 2, 364–374,.
16. P. Harikrishnan, B. L. F. Guillen, and K. Ravindran, , "Accretive operators and Banach Alaoglu ´ theorem in Linear 2-normed spaces," Proyecciones (Antofagasta), (2011), 30, 3,319–327.
17. M.Osilike and S. Aniagbosor, "Weak and strong convergence theorems for fixed points of asymptotically non expensive mappings," Mathematical and Computer Modelling,(2000),32, 10, 1181– 1191.
18. H.H. Bauschke and P.L.Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, second edition, Springe,.(2017)
19. H.H. Bauschke J.Y. Bello Cruz, T.T.A. Naghia H.M. Phan, and X. Wang, The rate of linear convergence of the Douglas-Rachford algorithm for subspaces is the cosine of the Friedric Journal of Approximation Theory (2014),185, 63–79.