FREE OSCILATION OF WATER IN A LAKE AND THE *I*-FUNCTION ASSOCIATED WITH THE M-SERIES

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Abstract- The object of this paper is to establish an infinite integral involving the product of M-series and the I-function. We have also obtained a solution of a problem of free oscillation of water in a circular lake with the help of the integral.

1. Introduction

The I-function which was introduced by Saxena [6] is an extension of Fox's H-function. On specializing the parameters this I-function reduces to Meijer's G-function, MacRobert's E-function, Lauricella function, Appell functions, Whittakar function etc. The importance of I-function lies in the fact that almost all the elementary and special functions in the literature follow as its special cases, therefore, the results established in this paper are of a general character and hence encompass several cases of interest.

We will represent and define here the I-function in the following manner:

$$I_{p_i,q_i:r}^{m,n}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{\mathcal{L}}\phi(s)x^s ds\ , \tag{1.1}$$

where

$$\phi(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\ \prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}s)\ \prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s)\right\}}, \tag{1.2}$$

$m,n,p_i(i=1,\dots,r);$ *and* $q_i(i=1,\dots,r)$ are integers satisfying $0\le n\le p_i\ ,1\le m\le q_i\ (i=1,\dots,r);r$ is finite $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji}$ are real and positive numbers $;a_j,b_j,a_{ji},b_{ji}$ are complex numbers such that

$$\alpha_j(\beta_h+\nu)\neq\beta_h(\alpha_j-\lambda-1)$$

for $\lambda,\nu=0,1,2,\dots\ ;h=1,2,\dots,m\,;i=1,2,\dots,r\,.$ $\mathcal{L}$ is a contour running from $\sigma-i\infty\ to\ \sigma+i\infty$ (σ is real), in the complex ξ - plane such that the points $s=\frac{(\alpha_j-\lambda-1)}{\alpha_j}$, $j=$

$1,2,\ldots n;\ \lambda=0,1,2,\ldots.$ and $\xi=\frac{(b_j+\lambda)}{\beta_j}$, $j=1,2,\ldots m;\ \lambda=0,1,2,\ldots$ lie to the left hand and right hand sides of $\mathcal{L}$ respectively. For the convergence conditions, existence of various contours L and other properties, we can refer Saxena [6, p. 35].

The well known M-series which is a particular case of H-function introduced by Inayat Hussain [3] and is defined by means of the following series expansion.

$$ {}_uM_v^{\tau}\left[\begin{smallmatrix}(A_1,\ldots,A_u);\\(B_1,\ldots,B_v);\end{smallmatrix} x\right]=\sum_{k=0}^{\infty}\frac{(A_1)_k\ldots(A_u)_k x^k}{(B_1)_k\ldots(B_v)_k\,\Gamma(\tau k+1)}, \tag{1.3}$$

provided that $\tau\epsilon c$, $R(\tau)>0$, $(A_j)_k(B_j)_k$ are pochhammer symbols.

2. Formula Required

The following formulae will be required in the present work:

From **Erdélyi** [2, p. 326, eq. (1)]:

$$\int_0^{\infty} x^{s-1}J_v(ax)dx=\frac{2^{s-1}\Gamma\left(\frac{s}{2}+\frac{\nu}{2}\right)}{a^s\Gamma\left(\frac{\nu}{2}-\frac{s}{2}+1\right)}\ ,\ -Re(\nu)<Re(s)<\frac{3}{2}. \tag{2.1}$$

From **Luke** [4, p. 291, (6)]:

$$\int_0^{\infty} x^{-1}J_{n+2m+1}(x)dx=\begin{cases}0,\ if\ m\neq n\\(4n+2a+2), if\ \ m=n\end{cases}. \tag{2.2}$$

3. An Infinite Integral

The integral to be evaluated is

$$\int_0^{\infty} x^{\lambda-1}J_v(cx)\ {}_uM_v^{\tau}\left[\begin{smallmatrix}(A_1,\ldots,A_u);\\(B_1,\ldots,B_v);\end{smallmatrix} hx^{2\rho}\right] I_{p_i,q_i:r}^{m,n}[zx^{-2\mu}]dx$$

$$=\frac{2^{\lambda-1}}{c^{\lambda}}\sum_{k=0}^{\infty}\left(\frac{2}{c}\right)^{2\rho k}\frac{(A_1)_k\ldots(A_u)_k\ h^k}{(B_1)_k\ldots(B_v)_k\Gamma(\tau\,k+1)}$$

$$\bullet\ I_{p_i,q_i+2;r}^{m+1,n}\left[z\left(\frac{2}{c}\right)^{-2\mu}\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\ \left(\frac{\lambda}{2}+\rho k+\frac{\nu}{2},\mu\right),(b_j,\beta_j)_{1,q},\left(\frac{\lambda}{2}+\rho k-\frac{\nu}{2},\mu\right)\end{matrix}\right.\right], \tag{3.1}$$

provided that λ and μ are positive integers, and $\delta>0,\Omega>0$, where

$$\Omega=\sum_{j=1}^{n}\alpha_j-\sum_{j=n+1}^{p_i}\alpha_{ji}+\sum_{j=1}^{m}\beta_j-\sum_{j=m+1}^{q_i}\beta_{ji}>0\,,\quad \delta=\sum_{j=1}^{q_i}\beta_{ji}-\sum_{j=1}^{p_i}\alpha_{ji}>0$$

Proof

To prove (3.1), in the LHS of equation (3.1) replacing M-series as given with equation (1.3) and I-function as Mellin-Barnes type contour integral (1.1), we find that

$$\int_0^\infty x^{\lambda-1}J_\nu(cx)\sum_{k=0}^{\infty}\frac{(A_1)_k\ldots(A_u)_k x^{2\rho k}h^k}{(B_1)_k\ldots(B_v)_k\Gamma(\tau k+1)}\left[\frac{1}{2\pi\omega}\int_{-\omega\infty}^{\omega\infty}\theta(\xi)z^{\xi}x^{-2\mu\xi}d\xi\right]dx. \qquad (3.2)$$

Interchanging the order of integration and summation, and evaluating the inner integral with the help of (2.1), we obtain

$$\sum_{k=0}^{\infty}\frac{(A_1)_k\ldots(A_u)_k\,h^k}{(B_1)_k\ldots(B_v)_k\Gamma(\tau k+1)}\left[\frac{1}{2\pi\omega}\int_{-\omega\infty}^{\omega\infty}\theta(\xi)z^{\xi}d\xi\right]$$

$$\bullet\frac{2^{\lambda+2\rho r-2\mu\xi-1}}{c^{\lambda+2\rho r-2\mu\xi-1}}\frac{\Gamma\left(\frac{\lambda}{2}+\rho r-\mu\xi+\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}-\frac{\lambda}{2}-\rho r+\mu\xi+1\right)}. \qquad (3.3)$$

Now applying (1.1), the RHS of the integral (3.1) is obtained.

4. Free Oscillation of Water in a Circular Lake

In this section product of M-series and the I–function has been employed in obtaining a solution of the following partial differential equation of oscillation of water in circular lake [5, p. 45 (I)]:

$$x^2\frac{\partial^2 y}{\partial x^2}+x\frac{\partial y}{\partial x}+\frac{\partial^2 y}{\partial\theta^2}-K^2x^2y=0, \qquad (4.1)$$

where y shows the depth of water surface from it position of equilibrium, $K=\frac{w}{\sqrt{(gd)}}$, the velocity of a free wave of small amplitude in a large expanse of water of depth d and g is the acceleration due to gravity with the following considerations

(1) In any vibrational mode y varies harmonically with time and y is small enough for its square to be neglected.
(2) The lake is stationary in space.
(3) There is no loss of energy.

5. Solution of the problem

The solution of (4.1) to be obtained here is

$$Y(x,\theta,t)=\left(\frac{2}{c}\right)^{\lambda}\sum_{i=0}^{\infty}\frac{i\,J_i(cx)\cos(i\theta-\phi_i)\cos(\omega it-\psi_i)}{\cos(\phi_i)\ \cos(\psi_i)}\sum_{k=0}^{\infty}\left(\frac{2}{a}\right)^{2\rho k}\frac{(A_1)_k\dots(A_u)_k}{(B_1)_k\dots(B_v)_k}$$

$$\bullet\frac{h^k}{\Gamma(\tau\,k+1)}I_{p_i,q_i+2;r}^{m+1,n}\left[z\left(\frac{2}{c}\right)^{-2\mu}\middle|\begin{matrix}(a_j,\alpha_j)_{1,p}\\ \left(\frac{\lambda}{2}+\rho k+\frac{\nu}{2},\mu\right),(b_j,\beta_j)_{1,q},\left(\frac{\lambda}{2}+\rho k-\frac{\nu}{2},\mu\right)\end{matrix}\right]. \quad (5.1)$$

The conditions of validity being the same as stated with (3.1).

PROOF

If we take into account all the physical assumptions [5, p. 47Art .2.61], the complete solution of (4.1) [5, p.47 (2)] is

$$Y(x,\theta,t)=\sum_{i=0}^{\infty}Q_iJ_i(cx)\cos(i\theta-\phi_i)\cos(\omega it-\psi_i) \quad (5.2)$$

where $\theta=t=0$.

Let

$$Y(x,0,0)=f(x)=x^{\lambda}\,{}_uM_v^{\tau}\left[\begin{matrix}(A_1,\dots,A_u);\\(B_1,\dots,B_v);\end{matrix}hx^{2\rho}\right]I_{p_i,q_i;r}^{m,n}[zx^{-2\mu}], \quad (5.3)$$

then

$$f(x)=x^{\lambda}\,{}_uM_v^{\tau}\left[\begin{matrix}(A_1,\dots,A_u);\\(B_1,\dots,B_v);\end{matrix}hx^{2\rho}\right]I_{p_i,q_i;r}^{m,n}[zx^{-2\mu}]=\sum_{i=0}^{\infty}Q_iJ_i(cx)\cos(\phi_i)\cos(\psi_i). \quad (5..4)$$

Multiplying both sides of (5.4) by $x^{-1}\,j_\nu(cx)$ integrating with respect to x from 0 to ∞ on the left hand side using (3.1) and on the right hand side using orthogonal property of Bessel function (2.2), we get

$$Q_\nu=\left(\frac{2}{c}\right)^{\lambda}\frac{\nu}{\cos(\phi_i)\cos(\psi_i)}\sum_{k=0}^{\infty}\left(\frac{2}{a}\right)^{2\rho k}\frac{(A_1)_k\dots(A_u)_k}{(B_1)_k\dots(B_v)_k}\frac{h^k}{\Gamma(\tau\,k+1)}$$

$$\bullet I_{p_i,q_i+2;r}^{m+1,n}\left[z\left(\frac{2}{c}\right)^{-2\mu}\middle|\begin{matrix}(a_j,\alpha_j)_{1,p}\\ \left(\frac{\lambda}{2}+\rho k+\frac{\nu}{2},\mu\right),(b_j,\beta_j)_{1,q},\left(\frac{\lambda}{2}+\rho k-\frac{\nu}{2},\mu\right)\end{matrix}\right]. \quad (5.5)$$

From (5.2) and (5.5), we obtain the desired result (5.1).

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