FUZZY SUB ALGEBRA AND FUZZY IDEALS OF GK ALGEBRA

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Abstract: **In this paper we introduce the fuzzification of GK algebra. We discuss about the fuzzy sub algebra of GK algebra and also fuzzy GK ideals of GK algebra and then we discuss about fuzzy Cartesian product of Fuzzy GK algebra and some interesting theorems.**

I.INTRODUCTION

The notion of fuzzy sets was introduced by L.A.Zadeh [5] and the notion of fuzzy group was introduced by Rosenfeld[3]. Later inspired by their results, O.G.Xi [4] introduced the notion of fuzzy BCK algebras. Afterwards Y.B.Jun and J.Meng [2] was studied fuzzy BCK algebra. Nowadays many authors have introduced the fuzzification of their work. In this paper we introduce the concept of fuzzy GK algebra.

II.FUZZY SUBALGERA OF GK ALGEBRA

***Definition:*2.1** A fu**zzy subset μ** of a GK algebra $(X,*,1)$ is called a fuzzy GK subalgebra of X, if the following conditions are satisfied

$\mu(x * y) \geq min\{\mu(x), \mu(y)\}$ for all x,y in X.

***Example*:2.2**

Consider X={1,2,3,4} is a GK algebra

*	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Define a mapping $\mu: X \rightarrow [0,1]$ by

$$\mu(x) = \begin{cases} 0.9 & if \;\; x = 1,2 \\ 0.5 & if \;\; otherwise \end{cases}$$

Then μ is a fuzzy GK subalgebra of X.

***Theorem*:2.3** Intersection of any two fuzzy GK subalgebras of X is again a fuzzy GK algebra.

Proof:

Let μ and δ be any two fuzzy GK subalgebras of X.Then,

$(\mu \cap \delta)(x * y) = \min\{\mu(x * y), \delta(x * y)\}$

$\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\delta(x), \delta(y)\}\}$

$= \min\{\min\{\mu(x), \delta(x)\}, \min\{\mu(y), \delta(y)\}\}$

$= \min\{(\mu \cap \delta)(x), (\mu \cap \delta)(y)\}$

$(\mu \cap \delta)(x * y) \geq \min\{(\mu \cap \delta)(x), (\mu \cap \delta)(y)\} \; \forall x, y \in X.$

Hence $\mu \cap \delta$ is fuzzy subalgebra of X.

***Definition:*2.4**

Let μ be any fuzzy subset of a GK algebra and let $s \in [0,1]$. The set $U(\mu, s) = \{x \in X: \mu(x) > s\}$ is called a level subset of μ in X.

Lemma:2.5

Let $(x, *, 1)$ be a GK algebra.Let μ be a fuzzy GK subalgebra of X. Let $\gamma \in [0,1]$.Then,

(i) $U(\mu, \gamma)$ is either $\emptyset$ or a GK subalgebra of X

(ii) $\mu(1) \geq \mu(x) \quad for \; all \;\; x \in X.$

Proof:

(i) For any $\gamma \in [0,1]$, assume that $U(\mu,\gamma)$ is non-empty.

Let $x, y \in U(\mu,\gamma)$. Then $\mu(x) \geq \gamma$ and $\mu(y) \geq \gamma$.

We need to prove $U(\mu,\gamma)$ is a GK subalgebra , for that we have to prove $x * y \in U(\mu,\gamma)$.

i.e., we need to prove $\mu(x * y) \geq \gamma$.

Now

$$\mu(x * y) \geq min\{\mu(x), \mu(y)\}$$

$$\geq \min\{\gamma, \gamma\} = \gamma$$

$\therefore\ \mu(x * y) \geq \gamma$

(ii) To prove $\mu(1) \geq \mu(x)$

$$\mu(1) = \mu(x * x)$$

$$\geq \min\{\mu(x), \mu(x)\} = \mu(x)$$

$Hence\ \ \mu(1) \geq \mu(x)\ \ for\ all\ \ x \in X.$

Theorem:2.6 If χ_1 and χ_2 are fuzzy GK subalgebras of X, then $\chi_1 \times \chi_2$ is a fuzzy GK algebra of $X \times X$.

Proof:

For any (x_1,x_2) and (y_1,y_2) $\in X \times X$.

Now,

$$\chi((x_1, x_2) * (y_1, y_2)) = \chi(x_1 * y_1, x_2 * y_2)$$

$$= (\chi_1 \times \chi_2)\ (x_1 * y_1, x_2 * y_2)$$

$$= \min\ \{(\chi_1(x_1 * y_1), \chi_2\ (x_2 * y_2)\}$$

$$\geq min\ \{min(\chi_1(x_1), \chi_1(y_1)),\ min(\chi_2(x_2), \chi_2(y_2))\}$$

$$= min\ \{min(\chi_1(x_1), \chi_2(x_2)),\ min(\chi_1(y_1), \chi_2(y_2))\}$$

$= \min\{ (\chi_1 \times \chi_2)\,(x_1, x_2),\ (\chi_1 \times \chi_2)(y_1 * y_2)\}$

$= \min\{\chi\,(x_1, x_2), \chi(y_1 * y_2)\}$

Hence χ is a fuzzy GK subalgebras of $X \times X$.

III.FUZZY GK IDEALS

***Definition*:3.1** Let X be a GK algebra.A fuzzy set $\mu\ in\ X$ is called fuzzy GK ideal of X if it satisfies the following conditions.

(i) $\mu(1) \geq \mu(x)$

(ii) $\mu(x * z) \geq \min\{\mu(y * z), \mu(y * x)\}\ \ \forall\, x, y, z \in X.$

***Example:*3.2** Consider the above example (2.2). This is an example of fuzzy GK ideal.

Theorem:3.3

Every fuzzy GK ideal of a GK-algebra X is order reversing.

Proof:

Let μ be a fuzzy GK ideal of a GK algebra X.

Let x, y$\in X$ be such that $x \leq y$ then $x * y = y * x = 1$.

Now, we know that $x * 1 = x$.

$\mu(x) = \mu(x * 1) \geq \min\{\mu(y * 1), \mu(y * x)\}$

$\geq \min\{\mu(y), \mu(1)\}$

$\geq \mu(y)$

Therefore μ is order reversing.

***Theorem*:3.4** If μ is a fuzzy ideal of GK algebra $(X,* 1)$ and $\mu_\gamma(x) = \min\{\gamma, \mu(x)\}\,\forall x \in X$ and $\gamma \in [0,1]$ then $\mu_\gamma(x)$ is fuzzy GK ideal of X.

Proof:

Let μ be a fuzzy ideal of GK algebra and $\gamma \in [0,1]$.

Therefore $\mu(1) \geq \mu(x) \; \forall \; x \in X$.

Now, $\mu_\gamma(1) = \min\{\gamma, \mu(1)\} \geq \min\{\gamma, \mu(x)\} = \mu_\gamma(x) \quad \forall x \in X$.

And we know that

$\mu(x * z) \geq \min\{\mu(y * z), \mu(y * x)\}$

Now

$\mu_\alpha(x * z) = \min\{\gamma, \mu(x * z)\}$

$\geq \min\{ \gamma, \min(\mu(y * z), \mu(y * x))\}$

$= \min \{ \min\left(\gamma, \mu(y * z)\right), \min(\gamma, \mu(y * x))\}$

$= \min\{\mu_\gamma(y * z), \mu_\gamma(y * x)\}$

Hence $\mu_\gamma(x)$ is fuzzy GK ideal of X.

***Proposition*:3.5**

Let μ be fuzzy GK ideal of GK algebra. If the inequality $y * x \leq z$ holds in X, then

$\mu(x) \geq \min\{\mu(y), \mu(z)\} \forall \; x, y, z \in X$.

Proof:

Assume that the inequality $y * x \leq z$ holds in X,

Then by theorem 3.3

$\mu(y * x) \geq \mu(z)$-------------(1)

By the definition fuzzy GK ideal

$\mu(x * z) \geq \min\{\mu(y * z), \mu(y * x)\}$

Put z=1

Then $\mu(x*1) \geq \min\{\mu(y*1), \mu(y*x)\}$

$\mu(x) \geq \min\{\mu(y), \mu(y*x)\}$ ------------(2)

From (1) and (2),

$\mu(x) \geq \min\{\mu(y), \mu(z)\}$.

Definition:3.6

Let μ *and* β be fuzzy subsets of a set S. The Cartesian product of μ *and* β is defined by

$(\mu \times \beta)(x, y) = \min\{\mu(x), \beta(y)\} \, \forall x, y \in S$

***Theorem*:3.7**

Let μ *and* β be fuzzy GK ideals of GK algebra X. Then $\mu \times \beta$ is a fuzzy GK ideal of $X \times X$.

Proof:

Let us consider

$(x, y) \in X \times X$

$(\mu \times \beta)(1,1) = \min\{\mu(1), \beta(1)\}$

$\geq \min\{\mu(x), \beta(y)\} = (\mu \times \beta)(x, y)$

Now let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$

$(\mu \times \beta)(x_1 * z_1, x_2 * z_2) = \min\{\mu(x_1 * z_1), \beta(x_2 * z_2)\}$

$\geq \min\{\min\{\mu(y_1 * z_1), \mu(y_1 * x_1)\}, \min\{\beta(y_2 * z_2), \beta(y_2 * x_2)\}\}$

$= \min\{\min\{\mu(y_1 * z_1), \beta(y_2 * z_2)\}, \min\{\mu(y_1 * x_1), \beta(y_2 * x_2)\}\}$

$= \min\{(\mu \times \beta)(y_1 * z_1, y_2 * z_2), (\mu \times \beta)(y_1 * x_1, y_2 * x_2)\}$

Therefore $\mu \times \beta$ is a fuzzy GK ideal of $X \times X$.

Theorem:3.8

Let $\mu\ and\ \beta$ be fuzzy subsets of GK algebra X such that $\mu \times \beta$ is a fuzzy GK ideal of $X \times X$.Then for all $x \in X$,

(i) Either $\mu(1) \geq \mu(x)\ or\ \beta(1) \geq \beta(x)$

(ii) $\mu(1) \geq \mu(x) \forall x \in X\ \ then\ either\ \beta(1) \geq \mu(x)\ or\ \beta(1) \geq \beta(x)$.

(iii) $If\ \beta(1) \geq \beta(x) \forall x \in X, then\ either\ \mu(1) \geq \mu(x)\ or\ \mu(1) \geq \beta(x)$.

(iv) $either\ \mu\ or\ \beta$ is a fuzzy GK ideal of X.

Proof:

(i) Suppose that $\mu(x) > \mu(1)\ and\ \beta(y) > \beta(1)$ for some $y \in X$.

Then

$(\mu \times \beta)(x, y) = \min\{\mu(x), \beta(y)\}$

$> \min\{\ \mu(1), \beta(1)\} = (\mu \times \beta\ )(1,1)$

This is a contradiction ,since $\mu \times \beta$ is a fuzzy GK ideal of $X \times X$.

Hence we obtain (i).

(ii) Assume that $x, y \in X$

$\mu(x) > \beta(1)\ and\ \beta(y) > \beta(1)$

Then we have

$(\mu \times \beta)(1,1) = \min\{\mu(1), \beta(1)\}$

$> \min\{\beta(1), \beta(1)\} = \beta(1)$

This implies that

$(\mu \times \beta)(x, y) = \min\{\mu(x), \beta(y)\}$

$> \min\{\beta(1), \beta(1)\} = \beta(1)$

$> (\mu \times \beta)(1,1)$

This is a contradiction.

Hence we obtain (ii)

(iii) By the similar way to part (ii)

(iv) In (i) we have

Either $(1) \geq \mu(x)$ or $\beta(1) \geq \beta(x) \forall x \in X$.

We assume that $\beta(1) \geq \beta(x)$, without loss of generality,

It is from (iii) such that

Either $\mu(1) \geq \mu(x)$ or $\mu(1) \geq \beta(x)$

If $\mu(1) \geq \beta(x)$ for any $x \in X$, then

$(\mu \times \beta)(1, x) = \min\{\mu(1), \beta(x)\} = \beta(x) ------(1)$

Now we have to prove β is a fuzzy GK ideal .

For that, let us consider $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X, we\ have$

Since $\mu \times \beta$ is a fuzzy GK ideal of $X \times X$, we have

$(\mu \times \beta)(x_1 * z_1, x_2 * z_2) \geq \min\{(\mu \times \beta)(y_1 * z_1, y_2 * z_2), (\mu \times \beta)(y_1 * x_1, y_2 * x_2)\}$

Now,if we take $x_1 = y_1 = z_1 = 1$, then

$(\mu \times \beta)(1, x_2 * z_2) \geq \min\{(\mu \times \beta)(1, y_2 * z_2), (\mu \times \beta)(1, y_2 * x_2)\}$

$Since$ by (1), LHS becomes

$\beta(x_2 * z_2) \geq \min\{(\mu \times \beta)(1, y_2 * z_2), (\mu \times \beta)(1, y_2 * x_2)\}$

$\geq \min\{\min\{\mu(1), \beta(y_2 * z_2)\}, \min\{\mu(1), \beta(y_2 * x_2)\}$

$\geq \min\{\beta(y_2 * z_2), \beta(y_2 * x_2)\}$

$\beta(x_2 * z_2) \geq \min\{\beta(y_2 * z_2), \beta(y_2 * x_2)\}$

This proves that β is a fuzzy GK ideal of X.

Now we consider $\mu(1) \geq \mu(x)$.

Suppose let us consider

$\mu(1) < \mu(y)$ for some y$\in X$

Then $\beta(1) \geq \beta(y) > \mu(1)$

Since $\mu(1) \geq \mu(x) \forall x \in X, then\ \beta(1) \geq \mu(x)$

Hence $(\mu \times \beta)(x, 1) = \min\{\mu(x), \beta(1)\} = \mu(x)$-------------(3)

Taking $x_2 = y_2 = z_2 = 1$ in (2)

$(\mu \times \beta)(x_1 * z_1, 1) \geq \min\{(\mu \times \beta)\,(y_1 * z_1, 1), (\mu \times \beta)(y_1 * x_1, 1)\}$

By (3)

$\mu(x_1 * z_1) \geq \min\{(\mu \times \beta)\,(y_1 * z_1, 1), (\mu \times \beta)(y_1 * x_1, 1)\}$

$\geq \min\{\,\min\{\mu((y_1 * z_1), \beta(\,1)\}, \min\{\mu(y_1 * x_1), \beta(1)\}$

$\geq \min\{\mu((y_1 * z_1), \mu(y_1 * x_1)\}$

$\mu(x_1 * z_1) \geq\ \min\{\mu((y_1 * z_1), \mu(y_1 * x_1)\}$

This proves that μ is a fuzzy GK ideal of GK algebra.

Therefore $either\ \mu\ or\ \beta$ is a fuzzy GK ideal of GK algebra X.

IV.CONCLUSION

In this paper we introduced the concept of fuzzy GK sub algebras of GK algebra. We discussed about fuzzy GK ideal and concept of Cartesian product of fuzzy GK algebra and some of the interesting results were also discussed.

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