ON $(1,2)^\star$-$r\hat{g}$-CLOSED SETS AND IT'S CHARACTERIZATIONS

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Abstract. In this paper is to introduce and study a new forms of generalized closed set namely $(1,2)^\star$-$r\hat{g}$-closed sets in bitopological spaces. This new class of closed set placed between $(1,2)^\star$-r-closed sets and $(1,2)^\star$-g-closed sets. We obtain several characterizations and some of their properties are investigate. Further, applying $(1,2)^\star$-$r\hat{g}$-closed sets to introduce a new class of space namely $(1,2)^\star$-$T_{r\hat{g}}$-space. Also investigate the relationship between this type of space and other existing spaces on the line of research.

1. Introduction

J.C.Kelly [1] was uttered the geometrical continuation of bitopological space that is a non empty set X together with two arbitrary topologies defined on X at the stage of significant study the shapes of objects. N. Levine [3] was initiated the study of generalizations of closed sets in topological spaces. M. Lellis Thivagar *et al.*, [4] established the properties of new type of bitopological open sets which are entirely different from Kellys pairwise open sets called $\tau_{1,2}$-open set and $\tau_1\tau_2$-open set. He [7] was also discussed the behaviour of $(1,2)^\star$-open set, $(1,2)^\star$-semi open set with their continuity and defined various types

[0]2010 Mathematics Subject Classification: 54A05, 54A10, 54C08, 54C10

Key words and phrases. $(1,2)^\star$-r-closed sets, $(1,2)^\star$-$\hat{g}$-closed sets, $(1,2)^\star$-$r\hat{g}$-closed sets and $(1,2)^\star$-$T_{r\hat{g}}$-spaces

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of bitopological generalized closed sets such as $(1,2)^\star$-sg-closed, $(1,2)^\star$-gs-closed sets and so on. In this paper is to introduce a new forms of closed sets called $(1,2)^\star$-$r\hat{g}$-closed in bitopological spaces. This new class of closed set placed between $(1,2)^\star$-r-closed sets and $(1,2)^\star$-g-closed sets. We obtain several characterizations and some of their properties are investigate. Further, applying $(1,2)^\star$-$r\hat{g}$-closed sets to introduce a new class of space namely $(1,2)^\star$-$T_{r\hat{g}}$-space. Also investigate the relationship between this type of space and other existing spaces on the line of research.

2. Preliminaries

Throughout this paper, (X,τ_1,τ_2) (or simply X) represent bitopological spaces (or simply space) on which no separation axioms are assumed unless otherwise mentioned. For a subset $A \subseteq X$, the closure and the interior of A are denoted by $\tau_{1,2}$-$cl(A)$ and $\tau_{1,2}$-$int(A)$, respectively.

The following basic concepts are using in this paper.

Definition 2.1. *Let S be a subset of X. Then S is said to be $\tau_{1,2}$-open* [5] *if $S = A \cup B$ where $A \in \tau_1$ and $B \in \tau_2$.*

The complement of $\tau_{1,2}$-open set is called $\tau_{1,2}$-closed.

Notice that $\tau_{1,2}$-open sets need not necessarily form a topology.

Definition 2.2. [5] *Let S be a subset of a bitopological space X. Then*

(1) *the $\tau_{1,2}$-closure of S, denoted by $\tau_{1,2}$-$cl(S)$, is defined as $\cap\{F : S \subseteq F$ and F is $\tau_{1,2}$-closed$\}$.*

(2) *the $\tau_{1,2}$-interior of S, denoted by $\tau_{1,2}$-$int(S)$, is defined as $\cup\{F : F \subseteq S$ and F is $\tau_{1,2}$-open$\}$.*

Definition 2.3. *A subset A of a bitopological space (X,τ_1,τ_2) or X is said to be*

(1) *a $(1,2)^\star$-semi open set* [5] *if $A \subseteq \tau_{1,2}$-$cl(\tau_{1,2}$-$int(A))$.*

(2) *a $(1,2)^\star$-α-open set* [5] *if $A \subseteq \tau_{1,2}$-$int(\tau_{1,2}$-$cl(\tau_{1,2}$-$int(A)))$.*

(3) *a* $(1,2)^{\star}$*-pre open set* [5] *if* $A \subseteq \tau_{1,2}$-$int(\tau_{1,2}$-$cl(A))$.

(4) *a* $(1,2)^{\star}$*-regular open set (briefly* $(1,2)^{\star}$*-r-open)* [6] *if* $A = \tau_{1,2}$-$int(\tau_{1,2}$-$cl(A))$.

The complement of the above mentioned set is called a closed set.

Definition 2.4. [8] *A subset A of a bitopological space* (X, τ_1, τ_2) *or X is said to be a* $(1,2)^{\star}$*-semi closure of A, denoted by* $(1,2)^{\star}$-$scl(A)$*, is defined as* $\cap\{F : S \subseteq F$ *and F is* $(1,2)^{\star}$*-semi closed*$\}$.

Definition 2.5. *A subset H of a bitopological space* (X, τ_1, τ_2) *or X is said to be*

(1) *a* $(1,2)^{\star}$*-generalized closed set (briefly* $(1,2)^{\star}$*-g-closed)* [9] *if* $\tau_{1,2}$-$cl(H) \subseteq U$ *whenever* $H \subseteq U$ *and U is* $\tau_{1,2}$*-open.*

(2) *a generalized* $(1,2)^{\star}$*-semi closed set (briefly* $(1,2)^{\star}$*-gs-closed)* [7] *if* $(1,2)^{\star}$-$scl(A) \subseteq U$ *whenever* $A \subseteq U$ *and U is* $\tau_{1,2}$*-open.*

(3) *a* $(1,2)^{\star}$*-semi generalized closed set (briefly* $(1,2)^{\star}$*-sg-closed)* [7] *if* $(1,2)^{\star}$-$scl(A) \subseteq U$ *whenever* $A \subseteq U$ *and U is* $(1,2)^{\star}$*-semi-open.*

(4) *an* α *generalized-closed set (briefly* $(1,2)^{\star}$*-*α*g-closed)* [10] *if* $(1,2)^{\star}$-$\alpha cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and U is* $\tau_{1,2}$*-open.*

(5) *a generalized* $(1,2)^{\star}$*-pre-closed set (briefly* $(1,2)^{\star}$*-gp-closed)* [5] *if* $(1,2)^{\star}$-$pcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and U is* $(1,2)^{\star}$*-open set.*

(6) *a generalized* $(1,2)^{\star}$*-*α*-closed set (briefly* $(1,2)^{\star}$*-g*α*-closed)* [10] *if* $(1,2)^{\star}$-$\alpha cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and U is* $(1,2)^{\star}$*-*α*- open set.*

(7) *a generalized* $(1,2)^{\star}$*-pre-closed set (briefly* $(1,2)^{\star}$*-gp-closed)* [5] *if* $(1,2)^{\star}$-$pcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and U is* $(1,2)^{\star}$*-open set.*

(8) *a* $(1,2)^{\star}$*-*$\hat{g}$*-closed set(=* $(1,2)^{\star}$*-*ω*-closed)* [2] *if* $\tau_{1,2}$-$cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and U is* $(1,2)^{\star}$*-semi open.*

The complements of the above mentioned closed sets are called their respective open sets.

Definition 2.6. *A bitopological space X is called, for a subset A is*

(1) $(1,2)^{\star}$-$T_{\frac{1}{2}}$-*space* [11] *if every* $(1,2)^{\star}$-*g-closed set in it is* $\tau_{1,2}$-*closed.*

(2) $(1,2)^{\star}$-T_b-*space* [11] *if every* $(1,2)^{\star}$-*gs-closed set in it is* $\tau_{1,2}$-*closed.*

(3) $(1,2)^{\star}$-$T_{\alpha\hat{g}}$-*space* [8] *if every* $(1,2)^{\star}$-α-*closed set in it is* $\tau_{1,2}$-*closed.*

(4) $(1,2)^{\star}$-αT_b-*space* [12] *if every* $(1,2)^{\star}$-αg-*closed set in it is* $\tau_{1,2}$-*closed.*

3. On $(1,2)^{\star}$-$r\hat{g}$-closed sets

Definition 3.1. *A subset A of a bitopological space* (X,τ_1,τ_2) *is called a* $(1,2)^{\star}$-$r\hat{g}$-*closed set if* $(1,2)^{\star}$-$rcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and U is* $\hat{g}$-*open. The compliment of* $(1,2)^{\star}$-$r\hat{g}$-*closed set is said to be* $(1,2)^{\star}$-$r\hat{g}$-*open.*

Theorem 3.2. *In a bitopological space* (X,τ_1,τ_2)*, every* $(1,2)^{\star}$-r-*closed is* $(1,2)^{\star}$-$r\hat{g}$-*closed.*

Proof. Let A be a $(1,2)^{\star}$-regular closed set and U be any $(1,2)^{\star}$-$\hat{g}$-open set containing A in (X,τ_1,τ_2). Since A is $(1,2)^{\star}$-regular closed, $\tau_{1,2}$-$cl(\tau_{1,2}$-$int(A)) = A$ For every subset A of X, which implies $(1,2)^{\star}$-$rcl(A) = A$. Therefore $(1,2)^{\star}$-$rcl(A) \subseteq U$. Thus A is $(1,2)^{\star}$-$r\hat{g}$-closed set.

Remark 3.3. *The reverse part of Theorem 3.2 is not true as seen from the following Example.*

Example 3.4. *Let* $X = \{a,b,c,d,e,f\}$ *be a non-empty set with* $\tau_1 = \{\phi,\{a\},X\}$ *and* $\tau_2 = \{\phi,\{a,c\},X\}$ *then* $\tau_{1,2} = \{\phi,\{a\},\{a,c\},X\}$. *In the space* (X,τ_1,τ_2)*, we have the subset* $\{a,b\}$ *is* $(1,2)^{\star}$-$r\hat{g}$-*closed set but not* $(1,2)^{\star}$-*regular closed.*

Theorem 3.5. *In a bitopological space* (X,τ_1,τ_2)*, every* $(1,2)^{\star}$-$r\hat{g}$-*closed set is* $(1,2)^{\star}$-g-*closed.*

Proof. Let A be a $(1,2)^{\star}$-$r\hat{g}$-closed set and U be any $\tau_{1,2}$-open set containing A in (X,τ_1,τ_2). Since every $\tau_{1,2}$-open set is $(1,2)^{\star}$-$\hat{g}$-open and A is $(1,2)^{\star}$-$r\hat{g}$-closed, $(1,2)^{\star}$-$rcl(A) \subseteq U$ for every subset A of X. Since $\tau_{1,2}$-$cl(A) \subseteq (1,2)^{\star}$-$rcl(A) \subseteq U$. Which implies $\tau_{1,2}$-$cl(A) \subseteq U$. Hence A is $(1,2)^{\star}$-g-closed.

Remark 3.6. *The reverse part of Theorem 3.5 is not true as seen from the following Example*

Example 3.7. *Let* $X = \{a, b, c\}$ *be a non-empty set with* $\tau_1 = \{\phi, \{a\}, X\}$ *and* $\tau_2 = \{\phi, \{b, c\}, X\}$ *then* $\tau_{1,2} = \{\phi, \{a\}, \{b, c\}, X\}$. *In the space* (X, τ_1, τ_2), *we have the subset* $\{a, b\}$ *is* $(1,2)^{\star}$-*g-closed but not* $(1,2)^{\star}$-$r\hat{g}$-*closed.*

Theorem 3.8. *In a bitopological space* (X, τ_1, τ_2), *every* $(1,2)^{\star}$-$r\hat{g}$-*closed set is* $(1,2)^{\star}$-*gs-closed.*

Proof. Let A be a $(1,2)^{\star}$-$r\hat{g}$-closed set and U be any $\tau_{1,2}$-open set containing A in (X, τ_1, τ_2). Since every $\tau_{1,2}$-open set is $(1, 2^{\star})$-$\hat{g}$-open and A is $(1,2)^{\star}$-$r\hat{g}$-closed, $(1,2)^{\star}$-$rcl(A) \subseteq U$ for every subset A of X. Since $(1,2)^{\star}$-$scl(A) \subseteq (1,2)^{\star}$-$rcl(A) \subseteq U$ which implies $\tau_{1,2}$-$cl(A) \subseteq U$. Thus A is $(1,2)^{\star}$-gs-closed.

Remark 3.9. *The reverse part of Theorem 3.8 is not true as seen from the following Example*

Example 3.10. *Let* $X = \{a, b, c\}$ *be a non-empty with* $\tau_1 = \{\phi, \{a\}, X\}$ *and* $\tau_2 = \{\phi, \{b\}, \{a, b\}, X\}$ *then* $\tau_{1,2} = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. *In the space* (X, τ_1, τ_2), *we have the subset* $\{a\}$ *is* $(1,2)^{\star}$-*gs-closed but not* $(1,2)^{\star}$-$r\hat{g}$-*closed.*

Remark 3.11. *The notions of* $(1,2)^{\star}$-$r\hat{g}$-*closed sets and the notions of* $(1,2)^{\star}$-*sg-closed sets are independent of a bitopolgical space* (X, τ_1, τ_2) *as seen from the following Examples.*

Example 3.12. *Let* $X = \{a, b, c\}$ *be a non-empty with* $\tau_1 = \{\phi, \{a\}, X\}$ *and* $\tau_2 = \{\phi, X\}$ *then* $\tau_{1,2} = \{\phi, \{a\}, X\}$. *In a space* (X, τ_1, τ_2), *we have the subset* $\{a, b\}$ *is* $(1,2)^{\star}$-$r\hat{g}$-*closed but not* $(1,2)^{\star}$-*sg-closed.*

Example 3.13. *Let* $X = \{a, b, c\}$ *be a non-empty with* $\tau_1 = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$ *and* $\tau_2 = \{\phi, \{a, b\}, X\}$ *then* $\tau_{1,2} = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. *In the space* (X, τ_1, τ_2), *we have the subset* $\{b\}$ *is* $(1,2)^{\star}$-*sg-closed but not* $(1,2)^{\star}$-$r\hat{g}$-*closed.*

Theorem 3.14. *In a bitopological space* (X, τ_1, τ_2)*, every* $(1,2)^\star$*-*$r\hat{g}$*-closed set is* $(1,2)^\star$*-*αg*-closed.*

Proof. Let A be a $(1,2)^\star$-$r\hat{g}$-closed set and U be any $\tau_{1,2}$-open set containing A in (X, τ_1, τ_2). Since every $\tau_{1,2}$-open set is $(1, 2^\star)$-$\hat{g}$-open, $(1,2)^\star$-$rcl(A) \subseteq U$ for every subset A of X. Since $(1,2)^\star$-$\alpha cl(A) \subseteq (1,2)^\star$-$rcl(A) \subseteq U$, which implies $(1,2)^\star$-$\alpha cl(A) \subseteq U$. Thus A is $(1,2)^\star$-αg-closed.

Remark 3.15. *The reverse part of Theorem 3.14 is not true as seen from the following Example.*

Example 3.16. *Let* $X = \{a, b, c\}$ *be a non-empty with* $\tau_1 = \{\phi, \{a, b\}, X\}$ *and* $\tau_2 = \{\phi, \{b\}, X\}$ *then* $\tau_{1,2} = \{\phi, \{b\}, \{a, b\}, X\}$*. In the space* (X, τ_1, τ_2)*, we have the subset* $\{a\}$ *is* $(1,2)^\star$*-*αg*-closed but not* $(1,2)^\star$*-*$r\hat{g}$*-closed.*

Remark 3.17. *The notions of* $(1,2)^\star$*-*$r\hat{g}$*-closed sets and the notions* $(1,2)^\star$*-*$g\alpha$*-closed sets are independent of a space* (X, τ_1, τ_2) *as seen from the following Examples.*

Example 3.18. *Let* $X = \{a, b, c\}$ *be a non-empty with* $\tau_1 = \{\phi, \{a\}, \{b, c\}, X\}$ *and* $\tau_2 = \{\phi, \{a\}, X\}$ *then* $\tau_{1,2} = \{\phi, \{a\}, \{b, c\}, X\}$*. In the space* (X, τ_1, τ_2)*, we have the subset* $\{a, b\}$ *is* $(1,2)^\star$*-*$g\alpha$*-closed but not* $(1,2)^\star$*-*$r\hat{g}$*-closed.*

Example 3.19. *Let* $X = \{a, b, c\}$ *be a non-empty with* $\tau_1 = \{\phi, \{b\}, X\}$ *and* $\tau_2 = \{\phi, X\}$ *then* $\tau_{1,2} = \{\phi, \{b\}, X\}$*. In the space* (X, τ_1, τ_2)*, we have the subset* $\{a, c\}$ *is* $(1,2)^\star$*-*$r\hat{g}$*-closed but not* $(1,2)^\star$*-*$g\alpha$*-closed.*

Theorem 3.20. *In a bitopological space* (X, τ_1, τ_2)*, every* $(1,2)^\star$*-*$r\hat{g}$*-closed set* $(1,2)^\star$*-*gp*-closed.*

Proof. Let A be a $(1,2)^\star$-$r\hat{g}$-closed set and U be any $\tau_{1,2}$-open set containing A in (X, τ_1, τ_2). Since every $\tau_{1,2}$-open set is $(1, 2^\star)$-$\hat{g}$-open, $(1,2)^\star$-$rcl(A) \subseteq U$ for every subset A of X. Since $(1,2)^\star$-$pcl(A) \subseteq (1,2)^\star$-$rcl(A) \subseteq U$ which implies $(1,2)^\star$-$pcl(A) \subseteq U$ and hence A is $(1,2)^\star$-gp-closed.

Remark 3.21. *The reverse part of Theorem 3.20 is not true as seen in the following Example.*

Example 3.22. *Let $X = \{a, b, c\}$ be a non-empty with $\tau_1 = \{\phi, \{a, b\}, X\}$ and $\tau_2 = \{\phi, X\}$ then $\tau_{1,2} = \{\phi, \{a, b\}, X\}$. In the space (X, τ_1, τ_2), we have the subset $\{a\}$ is $(1,2)^{\star}$-gp-closed but not $(1,2)^{\star}$-$r\hat{g}$-closed.*

Remark 3.23. *Obtain in the following Figure-I.*

$$
\begin{array}{ccccc}
 & & (1,2)^{\star}\text{-}gp\textbf{-closed} & & \\
 & & \Uparrow & & \\
(1,2)^{\star}\text{-}r\textbf{-closed} & \Rightarrow & (1,2)^{\star}\text{-}r\hat{g}\textbf{-closed} & \Rightarrow & (1,2)^{\star}\text{-}g\textbf{-closed} \\
 & & \Downarrow & & \\
 & & (1,2)^{\star}\text{-}gs\textbf{-closed} & &
\end{array}
$$

Remark 3.24. *Obtain in the following Figure-II.*

$$
(1,2^{\star})\text{-}\hat{g}\textbf{-closed} \quad \not\Leftrightarrow \quad (1,2)^{\star}\text{-}r\hat{g}\textbf{-closed} \quad \not\Leftrightarrow \quad (1,2)^{\star}\text{-}\alpha\textbf{-closed}
$$

Where $A \not\Leftrightarrow B$ represents A and B are independent of each other.

4. Other characterizations

Theorem 4.1. *The union two $(1,2)^{\star}$-$r\hat{g}$-closed subsets is always $(1,2)^{\star}$-$r\hat{g}$-closed of a space (X, τ_1, τ_2).*

Proof. Assuming that A and B are two $(1,2)^{\star}$-$r\hat{g}$-closed subset of X. Let U be a $(1,2^{\star})$-$\hat{g}$-open set in X such that $A \cup B \subseteq U$, then $A \subseteq U$ and $B \subseteq U$. Since A and B are $(1,2)^{\star}$-$r\hat{g}$-closed subset of X, $(1,2)^{\star}$-$rcl(A) \subseteq U$ and $(1,2)^{\star}$-$rcl(B) \subseteq U$, but $(1,2)^{\star}$-$rcl(A \cup B) \subseteq (1,2)^{\star}$-$rcl(A) \cup (1,2)^{\star}$-$rcl(B) \subseteq U$. Hence $A \cup B$ is also a $(1,2)^{\star}$-$r\hat{g}$-closed subset of X.

Remark 4.2. *The finite intersection of $(1,2)^{\star}$-$r\hat{g}$-closed sets but not $(1,2)^{\star}$-$r\hat{g}$-closed of a space (X, τ_1, τ_2) as seen from the following Example.*

Example 4.3. *Let $X = \{a, b, c\}$ be a non-empty set with $\tau_1 = \{\phi, \{c\}, X\}$ and $\tau_2 = \{\phi, X\}$ then $\tau_{1,2} = \{\phi, \{c\}, X\}$. In the space (X, τ_1, τ_2), we have the subsets $C = \{a, c\}$ and $D = \{b, c\}$ are $(1, 2)^\star$-$r\hat{g}$-closed sets but $E = C \cap D = \{c\}$ is not $(1, 2)^\star$-$r\hat{g}$-closed.*

Theorem 4.4. *Let A be a $(1, 2)^\star$-$r\hat{g}$-closed set of (X, τ_1, τ_2). Then $(1, 2)^\star$-$rcl(A) - A$ does not contain a non-empty $(1, 2^\star)$-$\hat{g}$-closed set.*

Proof. Suppose that A is $(1, 2)^\star$-$r\hat{g}$-closed. Let H be non-empty $(1, 2^\star)$-$\hat{g}$-closed set contained in $(1, 2)^\star$-$rcl(A) - A$. Now H^c is $(1, 2)^\star$-$\hat{g}$-open set of (X, τ_1, τ_2) such that $A \subseteq H^c$. Since A is $(1, 2)^\star$-$r\hat{g}$-closed set of (X, τ_1, τ_2), $(1, 2)^\star$-$rcl(A) \subseteq H^c$. Thus $G \subseteq ((1, 2)^\star$-$rcl(A))^c$. Also $G \subseteq (1, 2)^\star$-$rcl(A) - A$. Therefore $H \subseteq ((1, 2)^\star$-$rcl(A))^c \cap (1, 2)^\star$-$rcl(A) = \phi$, Hence $H = \phi$. Which is contradiction to the assumption $H \neq \phi$. Hence $(1, 2)^\star$-$rcl(A) - A$ does not contain a non-empty $(1, 2^\star)$-$\hat{g}$-closed set.

Theorem 4.5. *For A subset of a bitopological space (X, τ_1, τ_2), if A is $(1, 2^\star)$-$\hat{g}$-open and $(1, 2)^\star$-$r\hat{g}$-closed subset then A is a $(1, 2)^\star$-r-closed.*

Proof. Since A is $(1, 2^\star)$-$\hat{g}$-open and $(1, 2)^\star$-$r\hat{g}$-closed subset of (X, τ_1, τ_2), $(1, 2)^\star$-$rcl(A) \subseteq A$. Hence A is $(1, 2)^\star$-r-closed.

Theorem 4.6. *In a bitopological space (X, τ_1, τ_2), the intersection of $(1, 2)^\star$-$r\hat{g}$-closed and $(1, 2)^\star$-r-closed is always $(1, 2)^\star$-$r\hat{g}$-closed.*

Proof. Let A be $(1, 2)^\star$-$r\hat{g}$-closed and let G be $(1, 2)^\star$-r-closed. If F is a $(1, 2^\star)$-$\hat{g}$-open set containing $A \cap G$, $A \cap G \subseteq F$, then $A \subseteq F \cup G^c$ and so $(1, 2)^\star$-$rcl(A) \subseteq F \cup G^c$. Now $(1, 2)^\star$-$rcl(A \cap G) \subseteq (1, 2)^\star$-$rcl(A) \cap G \subseteq F$. Thus $A \cap G$ is $(1, 2)^\star$-$r\hat{g}$-closed.

Theorem 4.7. *If A is $(1, 2)^\star$-$r\hat{g}$-closed and $A \subseteq B \subseteq (1, 2)^\star$-$rcl(A)$ of a space (X, τ_1, τ_2), then B is also a $(1, 2)^\star$-$r\hat{g}$-closed of a space (X, τ_1, τ_2).*

Proof. Let H be an $(1,2^{\star})$-$\hat{g}$-open set of (X,τ_1,τ_2) such that $B \subseteq H$. Then $A \subseteq H$, since A is $(1,2)^{\star}$-$r\hat{g}$-closed set, $(1,2)^{\star}$-$rcl(A) \subseteq U$. Also since $B \subseteq (1,2)^{\star}$-$rcl(A)$, $(1,2)^{\star}$-$rcl(B) \subseteq (1,2)^{\star}$-$rcl((1,2)^{\star}$-$rcl(A)) = (1,2)^{\star}$-$rcl(A)$. Hence $(1,2)^{\star}$-$rcl(B) \subseteq H$. Therefore B is also a $(1,2)^{\star}$-$r\hat{g}$-closed set.

5. Applications

Definition 5.1. *A bitopological space (X,τ_1,τ_2) is called $(1,2)^{\star}$-$T_{r\hat{g}}$-space if every $(1,2)^{\star}$-$r\hat{g}$-closed set is $(1,2)^{\star}$-r-closed.*

Theorem 5.2. *For a bitopological space (X,τ_1,τ_2) the following conditions are equivalent:*

(1) *(X,τ_1,τ_2) is a $(1,2)^{\star}$-$T_{r\hat{g}}$-space.*

(2) *Every singleton $\{x\}$ is either $(1,2^{\star})$-$\hat{g}$-closed or $(1,2)^{\star}$-regular open.*

Proof. (1) $\Rightarrow$ (2) Let $x \in X$, Suppose $\{x\}$ is not a $(1,2^{\star})$-$\hat{g}$-closed set of (X,τ_1,τ_2). Then $X - \{x\}$ is not a $(1,2^{\star})$-$\hat{g}$-open set. Thus $X - \{x\}$ is a $(1,2)^{\star}$-$r\hat{g}$-closed set of (X,τ_1,τ_2). Since (X,τ_1,τ_2) is $(1,2)^{\star}$-$T_{r\hat{g}}$-space , $X - \{x\}$ is a $(1,2)^{\star}$-r-closed set of (X,τ_1,τ_2). That is $\{x\}$ is $(1,2)^{\star}$-r-open set of (X,τ_1,τ_2).

(2) $\Rightarrow$ (1) Let A be a $(1,2)^{\star}$-$r\hat{g}$-closed set of (X,τ_1,τ_2). Let $x \in \tau_{1,2}$-$rcl(A)$, By (2) $\{x\}$ is either $(1,2^{\star})$-$\hat{g}$-closed or $(1,2)^{\star}$-$r\hat{g}$-open.

Case(i): Let $\{x\}$ be $(1,2^{\star})$-$\hat{g}$-closed. If we assume that $x \in A$, then we would have $x \in \tau_{1,2}$-$rcl(A) - A$. Hence $x \in A$.

Case (ii): Let $\{x\}$ be $(1,2)^{\star}$-r-open, since $x \in (1,2)^{\star}$-$rcl(A)$. $\{x\} \cap A \neq \phi$. This shows that $x \in A$. So in the both cases we have $(1,2)^{\star}$-$rcl(A) \subseteq A$. Trivially $A \subseteq (1,2)^{\star}$-$rcl(A)$. Therefore $A = (1,2)^{\star}$-$rcl(A)$, which implies A is $(1,2)^{\star}$-r-closed. Hence (X,τ_1,τ_2) is $(1,2)^{\star}$-$T_{r\hat{g}}$-space.

Theorem 5.3. *Every $(1,2)^{\star}$-$T_{r\hat{g}}$-space is $(1,2)^{\star}$-$T_{\alpha\hat{g}}$-space.*

Proof. : Let (X,τ_1,τ_2) be a $(1,2)^\star$-$T_{r\hat{g}}$-space . Then every singleton is either $(1,2^\star)$-$\hat{g}$-closed or $(1,2)^\star$-r open. Since every $(1,2)^\star$-r-open is $(1,2)^\star$-α-open, every singleton is either $(1,2^\star)$-$\hat{g}$-closed or $(1,2)^\star$-α-open, Hence (X,τ_1,τ_2) is a $(1,2)^\star$-$T_{r\hat{g}}$-space.

Remark 5.4. *The reverse part of Theorem 5.3 is not true as seen from the following Example.*

Example 5.5. *Let* $X = \{a,b,c\}$ *be a non-empty set with* $\tau_1 = \{\phi,\{a\},\{a,b\},X\}$ *and* $\tau_2 = \{\phi,\{b\},\{a,b\},X\}$ *then* $\tau_{1,2} = \{\phi,\{a\},\{b\},\{a,b\},X\}$. *We have* $(1,2)^\star$-$T_{\alpha\hat{g}}$*-space but not* $(1,2)^\star$-$T_{r\hat{g}}$*-space.*

Remark 5.6. *The* $(1,2)^\star$-$T_{\frac{1}{2}}$*-space and* $(1,2)^\star$-$T_{r\hat{g}}$*-space are independent as seen from the following Examples.*

Example 5.7. *Let* $X = \{a,b,c\}$ *be a non-empty set with* $\tau_1 = \{\phi,\{a\},\{a,c\},X\}$ *and* $\tau_2 = \{\phi,\{c\},\{a,c\},X\}$ *then* $\tau_{1,2} = \{\phi,\{a\},\{c\},\{a,c\},X\}$. *We have* (X,τ_1,τ_2) *is a* $(1,2)^\star$-$T_{\frac{1}{2}}$*-space but not* $(1,2)^\star$-$T_{r\hat{g}}$*-space.*

Example 5.8. *Let* $X = \{a,b,c\}$ *be a non-empty set with* $\tau_1 = \{\phi,\{a\},X\}$ *and* $\tau_2 = \{\phi,\{b,c\},X\}$ *then* $\tau_{1,2} = \{\phi,\{a\},\{b,c\},X\}$. *We have* (X,τ_1,τ_2) *is a* $(1,2)^\star$-$T_{r\hat{g}}$*-space but not* $(1,2)^\star$-$T_{\frac{1}{2}}$*-space.*

Remark 5.9. *The* $(1,2)^\star$-$T_{r\hat{g}}$*-space is independent of* $(1,2)^\star$-T_b*-space as seen from the following Examples.*

Example 5.10. *Let* $X = \{a,b,c\}$ *be a non-empty set with* $\tau_1 = \{\phi,\{a\},\{a,b\},\{a,c\},X\}$ *and* $\tau_2 = \{\phi,X\}$ *then* $\tau_{1,2} = \{\phi,\{a\},\{a,b\},\{a,c\},X\}$. *We have* (X,τ_1,τ_2) *is* $(1,2)^\star$-T_b*-space but not* $(1,2)^\star$-$T_{r\hat{g}}$*-space.*

Example 5.11. *Let* $X = \{a,b,c\}$ *be a non-empty set with* $\tau_1 = \{\phi,X\}$ *and* $\tau_2 = \{\phi,\{b\},\{a,c\},X\}$ *then* $\tau_{1,2} = \{\phi,\{b\},\{a,c\},X\}$. *We have* (X,τ_1,τ_2) *is* $(1,2)^\star$-$T_{r\hat{g}}$*-space but not* $(1,2)^\star$-T_b*-space.*

Remark 5.12. *Obtain in the following Figure-III.*

$$\begin{array}{ccccc} & & (1,2)^{\star}\text{-}T_{\alpha\hat{g}}\textbf{-space} & & \\ & & \uparrow & & \\ (1,2)^{\star}\text{-}T_{\frac{1}{2}}\textbf{-space} & \not\Leftrightarrow & (1,2)^{\star}\text{-}T_{r\hat{g}}\textbf{-space} & \not\Leftrightarrow & (1,2)^{\star}\text{-}T_{b}\textbf{-space} \end{array}$$

(1) Where $A \to B$ represents A implies B but not converse.

(2) Where $A \not\Leftrightarrow B$ represents A and B are independent.

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