Fractional integral Operators Associated with Mellin and Laplace Transformations associated with *I*-Function

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Abstract *The object of this paper is to establish certain representations between the Laplace transform operators* L *and* L^{-1} *and the frectional integration operators due to Saigo and Maeda [1]. While two theorems on the fractional integration operators where also defined and studied earlier by them. And earlier the result proved by Ram, Saigo and Saxena[2] and Fox [3,4] are derived as special cases.*

Key words : Fractional calculus operators, Saigo-Maeda operators, Laplace transform , Mellin transform.

1. Introduction

Saigo [5] introduced an extension of both Riemann-Liouville and Erdélyi-kober fractional integration operators in terms of Gauss's hypergeometric function. Saxena [6], Kalla and Saxena [7], and Saxena and Kumbhat [8,9] had defined and studied earlier the fractional integration operators associated with gauss's hypergeometric functions.

Fox [3,4] investigated a representation of Erdélyi-kober operators in terms of Laplace transform operators L and L^{-1}.certain relations connecting L, L^{-1} and fractional operators of Saxena [10] were derived by Kumbhat and Saxena [11] and Saigo, Saxena and Ram [2], there by the results of Fox [3,4].

Saxena [12] defined a I-function which is introduced in the following form:

$$I^{m,n}_{p_i,q_i:r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{\mathcal{L}}\phi(s)x^s ds\ , \tag{1.1}$$

where

$$\phi(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j s)\ \prod_{j=1}^{n}\Gamma(1 - a_j + \alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1 - b_{ji} + \beta_{ji} s)\ \prod_{j=n+1}^{p_i}\Gamma(a_{ji} - \alpha_{ji} s)\right\}}, \tag{1.2}$$

$m, n, p_i (i = 1, \dots, r);\ and\ q_i (i = 1, \dots, r)$ are integers satisfying $0 \le n \le p_i$, $1 \le m \le q_i\ (i = 1, \dots, r)$; r is finite $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji}$ are real and positive numbers ; a_j, b_j, a_{ji}, b_{ji} are complex numbers such that

$$\alpha_j(\beta_h + \nu) \neq \beta_h(\alpha_j - \lambda - 1)$$

for $\lambda, \nu = 0,1,2, \dots\ ; h = 1,2, \dots, m\ ; i = 1,2, \dots, r$. $\mathcal{L}$ is a contour running from $\sigma - i\infty\ to\ \sigma + i\infty$ (σ is real), in the complex ξ - plane such that the points $s = \frac{(\alpha_j - \lambda - 1)}{\alpha_j}$, $j = 1,2, \dots n$; $\lambda = 0,1,2, \dots$ and $\xi = \frac{(b_j + \lambda)}{\beta_j}$, $j = 1,2, \dots m$; $\lambda = 0,1,2, \dots$ lie to the left hand and right hand sides of $\mathcal{L}$ respectively.

2. The Mellin and Laplace Transforms

The Mellin transform of $g(x)$ is defined by

$$M\{g(x): p\} = G(p) = \int_0^\infty x^{p-1} g(x)\,dx, \tag{2.1}$$

and the inverse Mellin transform is given by

$$g(x) = \frac{1}{2\pi i}\int_C x^{-p} G(p)\,dp, \tag{2.2}$$

where C is a suitable contour and p is a complex variable. The Parseval theorem for the Mellin transform are in the form

$$\int_0^\infty g(x)h(x)\,dx = \frac{1}{2\pi i}\int_C G(p)H(1-p)\,dp \tag{2.3}$$

where $G(p)$ and $H(p)$ are the Mellin transform of $g(x)$ and $h(x)$ respectively.

The laplace transform of a function is denoted by L and defined as

$$L\{f(x);p\} = F(p) = \int_0^\infty e^{-xp} f(x)dx \tag{2.4}$$

where $Re(p) > 0$.

The invers laplace transform of a function $F(p)$ is $f(x)$. The invers laplace transform is denoted by L^{-1} and defined as

$$L^{-1}\{F(p);x\} = f(x). \tag{2.5}$$

And the relation between L and L^{-1} is represented as

$$LL^{-1} = L^{-1}L = 1. \tag{2.6}$$

3. Fractional operators

In this section we show the definition of generalized fractional integration operators of arbitrary order involving Appells Functions due to Saigo and Maedo [1, p.393, eq. 4.12] in the kernel in the following form:

Let $a, a', b, b', c \in \mathbb{C}$ and $x > 0$, then the generalized fractional integration operators involving Appeall Functions F_3 are defined by the equations:

$$\left(I_{0,+}^{a,a',b,b',c} f\right)(x) = \frac{x^{-a}}{\Gamma(c)} \int_0^x (x-t)^{c-1} t^{-a'} F_3\left(a, a', b, b', c; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)dt, \tag{3.1}$$

where $Re(c) > 0$,

$$= \frac{d^l}{dx^l} I_{0,+}^{a,a',b+l,b',c+l} f, \tag{3.2}$$

where $Re(c) \leqq 0, l = \{-Re(c)\} + 1$,

and

$$\left(I_{-}^{a,a',b,b',c} f\right)(x) = \frac{x^{-a'}}{\Gamma(c)} \int_0^\infty (t-x)^{c-1} t^{-a} F_3\left(a, a', b, b', c; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)dt, \tag{3.3}$$

where $Re(c) > 0$,

$$= (-1)^l \frac{d^l}{dx^l} I_{0,+}^{a,a',b,b'+l,c+l} f, \tag{3.4}$$

where $Re(c) \leqq 0, l = \{-Re(c)\} + 1$.

For $a' = 0$ above operators reduces to Sigo operators [5], defined as:

Let $x \in \mathbb{R}_+ = (0, \infty)$ and a, b and η are complex numbers. The fractional operator $\{Re(a) > 0\}$ of function $f(x)$ on $\mathbb{R}_+$are defined as following to Saigo [5] as:

$$I_{0,+}^{a,b,\eta} f = \frac{x^{-a}}{\Gamma(a)} \int_0^x (x-t)^{a-1} \; {}_2F_1\left(a+b, -\eta; a; 1 - \frac{t}{x}\right) f(t) dt, \quad (3.5)$$

where $Re(a) > 0$,

$$= \frac{d^l}{dx^l} I_{0,+}^{a+l,b-l,\eta-l} f, \quad (3.6)$$

where $Re(a) \leqq 0, l = \{Re(-a)\} + 1$,

and

$$I_{-}^{a,b,\eta} f = \frac{1}{\Gamma(a)} \int_0^\infty (t-x)^{a-1} t^{-a-b} \; {}_2F_1\left(a+b, -\eta; a; 1 - \frac{x}{t}\right) f(t) dt, \quad (3.7)$$

where $Re(a) > 0$,

$$= (-1)^l \frac{d^l}{dx^l} I_{-}^{a+l,b-l,\eta} f, \quad (3.8)$$

where $Re(a) \leqq 0, l = \{Re(-a)\} + 1$.

4. Mellin Transforms of Fractional Calculus Operators

In this section we defined Mellin transforms of the fractional calculus operators $I_{0,+}^{a,a',b,b',c}$ and $I_{-}^{a,a',b,b',c}$.

Definition: Let $L_p(\mathbb{R}_+)$ be the usual Lebesgue class on $\mathbb{R}_+$ with $1 \leqq p < \infty$. We define $M_p(\mathbb{R}_+)$ as the class of all functions in $f \in L_p(\mathbb{R}_+)$ with $p > 2$ which are inverse Mellin transforms of the functions $L_p(\mathbb{R}_+)$, where $q = p/(p-1)$.

Theorem 1: Let $1 \leq \lambda \leq 2$ and $a, a', b, b', c \in \mathbb{C}$ with $Re(c) > 0$, satisfy

$Re(c) < 1 + \min[0, Re(b' - a-), Re(c - a - a' - b)]$, then for $f \in L_p(\mathbb{R}_+)$, the following formula holds.

$$M\left\{x^{a+a'-c}\left(I_{0,+}^{a,a',b,b',c} f\right)(x); k\right\} = \Gamma\begin{bmatrix} 1-k, 1-a'+b'-k, 1-a-a'-b+c-k \\ 1+b', 1-a-a'+c-k, 1-a'-b+c-k \end{bmatrix} M[f(x); k]. \quad (4.1)$$

Theorem 2: Let $1 \le \lambda \le 2$ and $a, a', b, b', c \in \mathbb{C}$ with $Re(c) > 0$, satisfy $Re(c) > \max\,[Re(-a-a'+c), Re(-a-b'+c), Re(b)]$, then for $f \in L_p(\mathbb{R}_+)$, the following formula holds.

$$M\{x^{a+a'-c}\left(I_{-}^{a,a',b,b',c}f\right)(x);k\} = \Gamma\left[{k+a+a'-c, k+a+b'-c, k-b \atop k+a+a'+b'-c, k, k+a-b}\right] M[f(x);k]. \quad (4.2)$$

5. Representation of Fractional Calculus Operators by Laplace Transform Operators

Theorem 3: Let $Re(c) > 0$, $Re(b'-a-) > 0$, $Re(c-a-a'-b) < 0$. If a function $f(x)$ satisfy the following conditions:

(I) $f(x) \in L(\mathbb{R}_+)$,

(II) $y^{-1/2}f(y) \in L(\mathbb{R}_+)$ where $f(y)$ is of bounded variation near to the point $y = x$,

(III) $M\{f(x);k\} = F(k) \in L\left(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty\right)$

(IV) $y^{b-1/2} I_{0,y}^{a,a',b,b',c} f \in L(\mathbb{R}_+)$ and $y^b I_{0,y}^{a,a',b,b',c} f$ is of bounded variation near the point $y = x$.

The following relation holds:

$$I_{0,+}^{a,a',b,b',c} f = x^{-a+b} L^{-1}\left[t^{a'+b-c} L\{x^{-b'} L^{-1}[t^{-a'} L\{x^{a-c+b'} L^{-1}[t^{-b} L\{x^{-a-a'-b+c} f(x)\}]\}]\}\right]. \quad (5.1)$$

Proof: By theorem for Saigo and Maeda [1] and from the condition (i) and (ii) we deduce that $x^{a+a'-c} I_{0,+}^{a,a',b,b',c} f$ exists on $(\mathbb{R}_+)$ and $f \in L_p(\mathbb{R}_+)$. thus the theorem 1. holds true. And the condition (iii) and theorem 28 of [13] yield

$$f(x) = \frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} F(k) x^{-k}\, dk. \quad (5.2)$$

Multiplying both side of above equation by $x^{-a-a'-b+c}$ and applying L operator, we have

$$L\{x^{-a-a'-b+c} f(x)\} = \int_0^\infty e^{-a}\, x^{-a-a'-b+c} \left\{\frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} F(k) x^{-k}\, dk\right\} dx. \quad (5.3)$$

The power of x is $Re(c - a - a' - b - {}^1/_2)$ along the line $k = {}^1/_2 + i\rho$. Thus the condition (i) and (iv) implys that the double integral in above equationis absolutely convbergent and we can change the order of integral to obtain

$$L\{x^{-a-a'-b+c}f(x)\} = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} F(k)\left[\int_0^\infty e^{-a}\, x^{-a-a'-b+c-k}dx\right]dk$$

$$= \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \Gamma(-a - a' - b + c - k + 1)t^{a+a'+b-c-1+k}F(k)dk. \quad (5.4)$$

Similarly multiply the above equation by t^{-b} and adopting the L^{-1} operator, we get

$$L^{-1}\left[t^{-b}L\{x^{-a-a'-b+c}f(x)\}\right]$$

$$= L^{-1}\left[\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \Gamma(-a - a' - b + c - k + 1)t^{-(1-a-a'+c-k)}F(k)dk\right]. \quad (5.5)$$

For applying theorem in Fox [3, p. 300], in above equation replacing $1 - k$ by k and applying [14], as

$$\Gamma(\mu + i\nu) = \sqrt{2\pi}|\nu|^{\mu-1/2}\exp\left(\frac{-\pi|\nu|}{2}\right) + O\left(\frac{1}{|\nu|}\right) \qquad (|\nu|) \to \infty$$

we find that

$$\frac{\Gamma(c - a - a' - b + k)}{\Gamma(c - a - a' + k)} = O\left(|k|^{-Re(b)}\right) \qquad |lm(k)| \to \infty.$$

Thus

$$\frac{\Gamma(c - a - a' - b + k)}{\Gamma(c - a - a' + k)}F(1 - k) \in L\left(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty\right),$$

For $Re(b) > 0$, the Fox's theorem implies that

$$L^{-1}\left[t^{-b}L\{x^{-a-a'-b+c}f(x)\}\right]$$

$$= \frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} \frac{\Gamma(c-a-a'-b+k)}{\Gamma(c-a-a'+k)} x^{-a-a'+c+k-1} F(1-k) dk. \quad (5.6)$$

Next multiplying the above equation by $x^{a-c+b'}$ and by using the L operator, this shows that

$$L\{x^{a-c+b'} L^{-1}[t^{-b} L\{x^{-a-a'-b+c} f(x)\}]\}$$

$$= \frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} \frac{\Gamma(c-a-a'-b+k)}{\Gamma(c-a-a'+k)} \Gamma(-a'+b'+k) \int_0^\infty t^{a'-b'-1} F(1-k) dk. \quad (5.7)$$

Similarly, multiply the above equation by $t^{-a'}$ and applying the operator L^{-1}, it yield

$$L^{-1}[t^{-a'} L\{x^{a-c+b'} L^{-1}[t^{-b} L\{x^{-a-a'-b+c} f(x)\}]\}]$$

$$= \frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} \frac{\Gamma(c-a-a'-b+k)}{\Gamma(c-a-a'+k)} \frac{\Gamma(-a'+b'+k)}{\Gamma(b'+k)} x^{b'+k-1} F(1-k) dk. \quad (5.8)$$

Similarly on multiplying the above equation by $t^{-b'}$ and on using the operator L, we obtain

$$L\{t^{-b'} L^{-1}[t^{-a'} L\{x^{a-c+b'} L^{-1}[t^{-b} L\{x^{-a-a'-b+c} f(x)\}]\}]\}$$

$$= \frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} \frac{\Gamma(c-a-a'-b+k)}{\Gamma(c-a-a'+k)} \frac{\Gamma(-a'+b'+k)}{\Gamma(b'+k)} \Gamma(k) t^{-k} F(1-k) dk. \quad (5.9)$$

Next on multiplying by $t^{a'+b-c}$ and adopting the operator L^{-1} to above equation, its gives

$$L^{-1}[t^{a'+b-c} L\{t^{-b'} L^{-1}[t^{-a'} L\{x^{a-c+b'} L^{-1}[t^{-b} L\{x^{-a-a'-b+c} f(x)\}]\}]\}]$$

$$= \frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} \frac{\Gamma(c-a-a'-b+k)\Gamma(-a'+b'+k)\Gamma(k)}{\Gamma(c-a-a'+k)\Gamma(b'+k)\Gamma(c-a'-b+k)} x^{-a'-b+c+k-1} F(1-k) dk$$

$$= \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \frac{\Gamma(1-c-a-a'-b-k)\Gamma(1-a'+b'-k)\Gamma(1-k)}{\Gamma(1+c-a-a'-k)\Gamma(1+b'-k)\Gamma(1+c-a'-b-k)}$$

$$x^{-a'-b+c-k}F(k)dk$$

$$= \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} M\left\{x^{a+a'-c}\left(I_{0,+}^{a,a',b,b',c}f\right)\right\}x^{-a'-b+c-k}F(k)dk. \quad (5.10)$$

By virtue of theorem 1, we reached at the required result

$$x^{-a+b}L^{-1}\left[t^{a'+b-c}L\left\{t^{-b'}L^{-1}\left[t^{-a'}L\left\{x^{a-c+b'}L^{-1}\left[t^{-b}L\left\{x^{-a-a'-b+c}f(x)\right\}\right]\right\}\right]\right\}\right]$$

$$= I_{0,+}^{a,a',b,b',c}f.$$

Hence this complete the proof.

Now to demonstrate the theorem 3, let us consider

$$f(x) = I_{p_i,q_i:r}^{m,n}\left[x\left|\begin{matrix}(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right]. \quad (5.11)$$

Then

$$L\left\{x^l I_{p_i,q_i:r}^{m,n}\left[x\left|\begin{matrix}(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right]\right\} = t^{-l-1}$$

$$I_{p_i+1,q_i:r}^{m,n+1}\left[t^{-1}\left|\begin{matrix}(-l-1),(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right]. \quad (5.12)$$

Provides that's $Re(t) > 0, \min_{1\le j\le m}\left[Re\left(\frac{b_j}{\beta_j}\right)\right] + Re(l) > -1$

We have

$$L\left\{x^{-a-a'-b+c} I_{p_i,q_i:r}^{m,n}\left[x\left|\begin{matrix}(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right]\right\} = t^{a+a'+b-c-1}$$

$$I^{m,n+1}_{p_i+1,q_i:r}\left[t^{-1}\left|\begin{matrix}(a+a'+b-c,1),(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right]. \quad (5.13)$$

Similarly on multiplying the above equation by $t^{-\beta}$ and then applying L^{-1}, we get

$$L^{-1}\left[t^{-b}L\left\{x^{-a-a'-b+c}I^{m,n}_{p_i,q_i:r}\left[x\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]\right\}\right]$$

$$=L^{-1}\left[t^{a+a'-c-1}I^{m,n+1}_{p_i+1,q_i:r}\left[t^{-1}\left|\begin{matrix}(a+a'+b-c,1),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]\right],$$

$$=x^{-a-a'+c}I^{m,n+1}_{p_i+1,q_i+1:r}\left[y\left|\begin{matrix}(a+a'+b-c,1),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},(a+a'-c,1)\end{matrix}\right.\right]. \quad (5.14)$$

Now by multiplying the above equation by $x^{a-c+b'}$ and then applying L operator, we get

$$L\left\{x^{a-c+b'}L^{-1}\left[t^{-b}L\left\{x^{-a-a'-b+c}I^{m,n}_{p_i,q_i:r}\left[x\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]\right\}\right]\right\}$$

$$=L\left\{x^{-a'+b'}I^{m,n+1}_{p_i+1,q_i+1:r}\left[x\left|\begin{matrix}(a+a'+b-c,1),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},(\alpha+\alpha'-\gamma,1)\end{matrix}\right.\right]\right\},$$

$$=t^{\alpha'-\beta'-1}I^{m,n+2}_{p_i+2,q_i+1:r}\left[t^{-1}\left|\begin{matrix}(a'-b',1)(a+a'+b-c,1),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},(a+a'-c,1)\end{matrix}\right.\right]. \quad (5.15)$$

Again multiplying it by $t^{-a'}$ and then applying L^{-1} operator to its, it yields that

$$L^{-1}\left[t^{-a'}L\left\{x^{a-c+b'}L^{-1}\left[t^{-b}L\left\{x^{-a-a'-b+c}I^{m,n}_{p_i,q_i:r}\left[x\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]\right\}\right]\right\}\right]$$

$$=L^{-1}\left[t^{-b'-1}I^{m,n+2}_{p_i+2,q_i+1:r}\left[t^{-1}\left|\begin{matrix}(a'-b',1)(a+a'+b-c,1),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},(a+a'-c,1)\end{matrix}\right.\right]\right],$$

$$= x^{b'} I^{m,n+2}_{p_i+2,q_i+2:r}\left[x\left|\begin{matrix}(\alpha'-\beta',1)(\alpha+\alpha'+\beta-\gamma,1),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},(\alpha+\alpha'-\gamma,1),(-\beta',1)\end{matrix}\right.\right]. \quad (5.16)$$

Next, multiplying the above equation by $x^{-b'}$ and by applying the L operator, we get

$$L\left\{x^{-b'}L^{-1}\left[t^{-a'}L\left\{x^{a-c+b'}L^{-1}\left[t^{-b}L\left\{y^{-a-a'-b+c}I^{m,n}_{p_i,q_i:r}\left[x\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]\right\}\right]\right\}\right]\right\}$$

$$= L\left\{I^{m,n+2}_{p_i+2,q_i+2:r}\left[x\left|\begin{matrix}(a'-b',1)(a+a'+b-c,1),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},(a+a'-c,1),(-b',1)\end{matrix}\right.\right]\right\},$$

$$= t^{-1}I^{m,n+3}_{p_i+3,q_i+2:r}\left[t^{-1}\left|\begin{matrix}(0,1),(a'-b',1),(a+a'+b-c,1),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},(a+a'-c,1),(-b',1)\end{matrix}\right.\right]. \quad (5.17)$$

Finally on multiplying the above equation by $t^{a'+b-c}$ and then readdress the operator L^{-1}, it show that

$$L^{-1}\left[t^{a'+b-c}L\left\{y^{-b'}L^{-1}\left[t^{-a'}L\left\{x^{a-c+b'}L^{-1}\left[t^{-b}L\left\{y^{-a-a'-b+c}\right.\right.\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left.\left.I^{m,n}_{p_i,q_i:r}\left[x\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]\right\}\right]\right\}\right]\right\}\right] = L^{-1}\left[t^{a'+b-c-1}\right.$$

$$\left.I^{m,n+3}_{p_i+3,q_i+2:r}\left[t^{-1}\left|\begin{matrix}(0,1),(a'-b',1),(a+a'+b-c,1),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},(a+a'-c,1),(-b',1)\end{matrix}\right.\right]\right],$$

$$= x^{-a'-b+c}I^{m,n+3}_{p_i+3,q_i+3:r}\left[x\left|\begin{matrix}(0,1),(a'-b',1),(a+a'+b-c,1),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},(a+a'-c,1),(-b',1),(a'+b-c,1)\end{matrix}\right.\right]. \quad (5.18)$$

The following result holds

$$I^{a,a',b,b',c}_{0,+}\left\{I^{m,n}_{p_i,q_i:r}\left[x\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]\right\} = x^{-a'-b+c}$$

$$I^{m,n+3}_{p_i+3,q_i+3:r}\left[x\left|\begin{matrix}(0,1),(a'-b',1),(a+a'+b-c,1),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},(a+a'-c,1),(-b',1),(a'+b-c,1)\end{matrix}\right.\right]. \quad (5.19)$$

Hence we get the result.

Theorem 4: Let $Re(c)>0$, $Re(-a-a'+c)>0$, $Re(-a-b'+c)>0$, and $Re(b)>0$. If a function $f(x)$ satisfy the following conditions:

(I) $f(x)\in L(\mathbb{R}_+)$,

(II) $y^{-1/2}f(y)\in L(\mathbb{R}_+)$ where $f(y)$ is of bounded variation near to the point $y=x$,

(III) $M\{f(x);k\}=F(k)\in L\left(\frac{1}{2}-i\infty,\frac{1}{2}+i\infty\right)$

(IV) $y^{b-1/2}I^{a,a',b,b',c}_{y,\infty}f\in L(\mathbb{R}_+)$ and $y^{b}I^{a,a',b,b',c}_{y,\infty}f$ is of bounded variation near the point $y=x$.

The following relation holds:

$$I^{a,a',b,b',c}_{-}f=x^{-2a-a'+b+c-1}L^{-1}\left[t^{b+b'-c}L\left\{x^{-a'}L^{-1}\left[t^{-b'}L\left\{x^{a+a'-c}L^{-1}\left[t^{-b}\right.\right.\right.\right.\right.$$

$$L\{x^{-b-1}f(x)\}]\}]\}]=\frac{1}{\chi}. \quad (5.20)$$

Proof: The theorem (5.2) can be established by following the steps of proof of theorem (5.1). For its prove replace x with $x^{-1}$and then apply L and L^{-1} operators. We introduce

$$\frac{\Gamma(k+a+a'-c)\Gamma(k+a+b'-c)\Gamma(k-b)}{\Gamma(k+a+a'+b'-c)\Gamma(k)\Gamma(k+a-b')}$$

in (5.5.10), from equation (5.4.2) we contains

$$M\left\{x^{a+a'-c}\left(I^{a,a',b,b',c}_{-}f\right)\right\}x^{a-b+k-1}$$

On removing x with $x^{-1}$and using (5.2), thuse equation (5.20) obtain.

On demonstrating theorem 4 in the same way, we can obtain

$$I^{a,a',b,b',c}_{-}\left\{I^{m,n}_{p_i,q_i:r}\left[x\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]\right\}=x^{-a+c}$$

$$I^{m+3,n}_{p_i+3,q_i+3:r}\left[x\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i},(0,1),(a-b,1),(a+a'+b'-c,1)\\(a+a'-c,1),(-b,1),(a+b'-c,1),(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]. \quad (5.5.21)$$

Special case

i) On taking $r = 1$ in equation (5.19), I-function reduces into H-function the result so obtain is a special case of a formula obtain by Saxena and Saigo [15, p. 94, eq. 3.2].

$$I^{a,a',b,b',c}_{0,+}\left\{H^{m,n}_{p,q}\left[x\left|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right.\right]\right\}$$
$$= x^{-a'-b+c}I^{m,n+3}_{p+3,q+3}\left[x\left|\begin{matrix}(0,1),(a'-b',1),(a+a'+b-c,1),(a_p,\alpha_p)\\(b_q,\beta_q),(a+a'-c,1),(-b',1),(a'+b-c,1)\end{matrix}\right.\right]$$

ii) On putting $r = 1$ in equation (5.21), I-function reduces into H-function the result so obtain is a special case of a formula obtain by Saxena and Saigo [15, p. 96, eq. 4.21].

$$I^{a,a',b,b',c}_{-}\left\{H^{m,n}_{p,q}\left[x\left|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right.\right]\right\} = x^{-a+c}$$

$$H^{m+3,n}_{p+3,q+3}\left[x\left|\begin{matrix}(a_p,\alpha_p),(0,1),(a-b,1),(a+a'+b'-c,1)\\(a+a'-c,1),(-b,1),(a+b'-c,1),(b_q,\beta_q)\end{matrix}\right.\right].$$

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