$\ddot{g}_p$-LOCALLY CLOSED SETS IN TOPOLOGICAL SPACES

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Abstract. In this paper is to introduce a new forms of some locally closed sets called $\ddot{g}_p$-locally closed sets, $\ddot{g}_p$-lc* sets and $\ddot{g}_p$-lc** sets. Properties of these new concepts are studied as well as their relations to the other classes of locally closed sets are investigated in topological spaces.

1. Introduction

The first step of locally closedness was done by N. Bourbaki [7]. He defined a set A to be locally closed if it is the intersection of an open set and a closed set in topological spaces. In literature many general authors introduced the studies of locally closed sets in topological spaces. Extensive research on locally closedness and generalizing locally closedness were done in recent years. A. H. Stone [22] used the term FG for a locally closed set. M. Ganster and Reilly used locally closed sets in [10] to define LC-continuity and LC-irresoluteness. K. Balachandran *et al.*, [4] introduced the concept of generalized locally closed sets. M. Veera Kumar [24, 27, 29] introduced $\hat{g}$-LC, g^*-LC and $g^{\#}$-LC sets and so on. In this paper is to introduce new forms of some locally closed sets called $\ddot{g}_p$-locally closed sets, $\ddot{g}_p$-lc* sets and $\ddot{g}_p$-lc** sets. Properties of these new concepts are studied as well as their relations to

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the other classes of locally closed sets are investigated in topological spaces. More over a comparative picture is given and many examples are given to show the converse are not true.

2. Preliminaries

Throughout this paper a topological space (X, τ) (or X) represents topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a topological space (X, τ), $cl(A)$, $int(A)$ and A^c denote the closure of A, interior of A and complement of A respectively in X.

We recall the following definitions which are useful in the sequel.

Definition 2.1. *A subset A of a topological space (X, τ) is called*

(1) *a semi-open set* [12] *if $A \subseteq cl(int(A))$.*
(2) *an α-open set* [18] *if $A \subseteq int(cl(int(A)))$.*
(3) *a pre-open set* [17] *if $A \subseteq int(cl(A))$.*
(4) *a semi-pre open set* [1] *(=β-open set) if $A \subseteq cl(int(cl(A)))$.*

The complement of above sets are called closed sets in X. The intersection of all semi closed (resp. pre-closed, semi-preclosed, α-closed) sets containing a subset A of (X, τ) is called the semi-closure (resp. pre-closure, semi-pre-closure and α-closure) of A and is denoted by $scl(A)$ (resp. $pcl(A)$, $spcl(A)$ and $\alpha cl(A)$).

Definition 2.2. *A subset A of a topological space (X, τ) is called*

(1) *a generalized closed set (briefly g-closed)* [13] *if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.*
(2) *a strongly generalized closed set (briefly g^*-closed)* [23] *if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open.*
(3) *a semi-generalized closed set (briefly sg-closed)* [6] *if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open.*

(4) *a generalized semi-closed set (briefly gs-closed)* [2] *if* $scl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open.*

(5) *a generalized* α*-closed set (briefly g*α*-closed)* [14] *if* $\alpha cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is* α*-open.*

(6) *an* α*-generalized closed set (briefly* α*g-closed)* [15] *if* $\alpha cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open.*

(7) *a generalized semi-pre closed set (briefly gsp-closed)* [8] *if* $spcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open.*

(8) *a generalized pre closed set (briefly gp-closed)* [16] *if* $pcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open.*

(9) *a generalized pre-regular closed set (briefly gpr-closed)* [11] *if* $pcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is regular open.*

(10) *a* $\hat{g}$*-closed set* [28] *if* $cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is semi open.*

(11) *a* g^*p*-closed set* [25] *if* $pcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open.*

(12) *a* $g^{\#}$*-closed set* [26] *if* $cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is* αg*-open.*

(13) *a* g^*s*-closed set* [19] *if* $scl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is gs-open.*

(14) *a* $\ddot{g}$*-closed set* [21] *if* $cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is sg-open.*

(15) *a* $\ddot{g}_p$*-closed set* [3] *if* $pcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is* $\ddot{g}$*-open.*

The complement of above sets are called an open sets in X.

Definition 2.3. *A subset* A *of a topological space* (X, τ) *is called*

(1) *a locally closed set (briefly lc) set* [10] *if* $A = U \cap F$*, where* U *is open and* F *is closed.*

(2) *a generalized locally closed set (briefly glc set)* [5] *if* $A = U \cap F$*, where* U *is g-open and* F *is g-closed.*

(3) *a* $\hat{g}$*-locally closed set (briefly* $\hat{g}lc$ *set)* [24] *if* $A = U \cap F$*, where* U *is* $\hat{g}$*-open and* F *is* $\hat{g}$*-closed.*

(4) *a $g^{\#}$-locally closed set (briefly $g^{\#}lc$ set)* [29] *if* $A = U \cap F$, *where* U *is* $g^{\#}$*-open and* F *is* $g^{\#}$*-closed.*

(5) *a g^*-locally closed set (briefly g^*lc set)* [27] *if* $A = U \cap F$, *where* U *is* g^**-open and* F *is* g^**-closed.*

(6) *a g^*s-locally closed set (briefly g^*slc set)* [20] *if* $A = U \cap F$, *where* U *is* g^*s*-open and* F *is* g^*s*-closed.*

Definition 2.4. *A topological space* (X, τ) *is called*

(1) *a sub maximal space* [9] *if every dense subset is open.*

(2) *a semi-pre-$T_{\frac{1}{2}}$ space* [6] *if every gsp-closed set is semi-pre closed.*

Definition 2.5. [3] *A subset* A *of a topological space* (X, τ) *is called*

(1) *a* $\ddot{g}_p$*-dense if* $\ddot{g}_p$*-cl*$(A) = X$.

(2) *a* $\ddot{g}_p$*-sub maximal if every dense subset in it is* $\ddot{g}_p$*-open.*

3. $\ddot{g}_p$-Locally closed sets

Definition 3.1. *A subset* A *of a topological space* (X, τ) *is called a* $\ddot{g}_p$*-locally closed set (briefly* $\ddot{g}_p$*-lc set) if* $A = E \cap F$, *where* E *is* $\ddot{g}_p$*-open and* F *is* $\ddot{g}_p$*-closed.*

The family of all $\ddot{g}_p$-locally closed sets in X is denoted by $\ddot{G}_P LC(X)$.

Definition 3.2. *A subset* A *of a topological space* (X, τ) *is called*

(1) $\ddot{g}_p$*-lc** *set if* $A = E \cap F$, *where* E *is* $\ddot{g}_p$*-open and* F *is closed.*

(2) $\ddot{g}_p$*-lc*** *set if* $A = E \cap F$, *where* E *is open and* F *is* $\ddot{g}_p$*-closed.*

The family of all $\ddot{g}_p$-lc* set (resp. $\ddot{g}_p$-lc** set) in X is denoted by $\ddot{G}_P LC^*(X)$ (resp. $\ddot{G}_P LC^{**}(X)$.

Proposition 3.3. *In a topological space* (X, τ), *every locally closed set is* $\ddot{g}_p$*-lc set.*

Proof. Let A be a locally closed set in (X, τ). Then there exist an open set E and closed set F such that $A = E \cap F$. Since every closed set is $\ddot{g}_p$-closed and, its complement is $\ddot{g}_p$-open. Thus A is $\ddot{g}_p$-lc set.

Remark 3.4. *The following Example shows that converse of Proposition 3.3 is not true.*

Example 3.5. *Let $X = \{1, 2, 3\}$ be a non-empty set and $\tau = \{\phi, \{1\}, X\}$, we have $\ddot{g}_p$-lc $= \wp(X)$. Clearly the subset $\{1, 2\}$ is $\ddot{g}_p$- locally closed set but not locally closed set in (X, τ).*

Proposition 3.6. *In a topological space (X, τ), every $\hat{g}$-lc set is $\ddot{g}_p$-lc set.*

Proof. Let A be $\hat{g}$-lc set in X. Then there exist an $\hat{g}$-open set E and $\hat{g}$-closed set F such that $A = E \cap F$. Since every $\hat{g}$-closed set is $\ddot{g}_p$-closed set, its complement is $\ddot{g}_p$-open. Thus A is $\ddot{g}_p$-lc set.

Remark 3.7. *The following Example shows that converse of Proposition 3.6 is not true.*

Example 3.8. *Let $X = \{1, 2, 3\}$ be a non-empty set and $\tau = \{\phi, \{1\}, \{1, 2\}, X\}$, we have $\ddot{g}_p$-lc $= \wp(X)$. Clearly the subset $\{2\}$ is $\ddot{g}_p$-locally closed but not $\hat{g}$-lc set.*

Proposition 3.9. *In a topological space (X, τ), every g^*-lc set is $\ddot{g}_p$-lc set.*

Proof. Let A be g^*-lc set in X. Then there exist an g^*-open set E and g^*-closed set F such that $A = E \cap F$. Since every g^*-closed set is $\ddot{g}_p$-closed, and its complement is g^*-open. Thus A is $\ddot{g}_p$-lc set.

Remark 3.10. *The following Example shows that converse of Proposition 3.9 is not true.*

Example 3.11. *In Example 3.5, we have $\ddot{g}_p$-lc $= \wp(X)$. Clearly the subset $\{3\}$ is $\ddot{g}_p$-lc set but not g^*-lc set.*

Proposition 3.12. *In a topological space (X, τ), every $g^{\#}$-lc set is $\ddot{g}_p$-lc set.*

Proof. Let A be $g^{\#}$-lc set. Then there exist an $g^{\#}$-open set E and $g^{\#}$-closed set F such that $A = E \cap F$. Since every $g^{\#}$-closed set is $\ddot{g}_p$-closed and its complement is $\ddot{g}_p$-open. Thus A is $\ddot{g}_p$- lc set.

Remark 3.13. *The following Example shows that converse of Proposition 3.12 is not true.*

Example 3.14. *Let $X = \{1, 2, 3\}$ be a non-empty set and $\tau = \{\phi, \{1\}, \{2, 3\}, X\}$. We have the subset $\{1, 2\}$ is $\ddot{g}_p$-lc set but not $g^{\#}$-lc set.*

Proposition 3.15. *In a topological space (X, τ), every locally closed set is $\ddot{g}_p$-lc* set.*

Proof. Let A be a locally closed set in X. Then there exist E and closed set F such that $A = E \cap F$. Since every open set is $\ddot{g}_p$-open. Thus A is $\ddot{g}_p$-lc* set.

Remark 3.16. *The following Example shows that converse of Proposition 3.15 is not true.*

Example 3.17. *Let $X = \{1, 2, 3\}$ be a non-empty set and $\tau = \{\phi, \{1\}, X\}$. We have the subset $\{a, c\}$ is $\ddot{g}_p$-lc* set but not locally closed.*

Proposition 3.18. *In a topological space (X, τ), every locally closed set is $\ddot{g}_p$-lc** set.*

Proof. Let A be locally closed set in X. Then there exist an open set E and closed set F such that $A = E \cap F$. Since every closed set is $\ddot{g}_p$-closed. Thus A is $\ddot{g}_p$-lc**.

Remark 3.19. *The following Example shows that converse of Proposition 3.18 is not true.*

Example 3.20. *In Example 3.17, then the subset $\{1, 3\}$ is $\ddot{g}_p$-lc** set but not locally closed.*

Theorem 3.21. *Let A and B be any two subsets of (X, τ), then*

(1) $A \in \ddot{G}_P LC(X)$ and B is $\ddot{g}_p$-open $\Rightarrow A \cap B \in \ddot{G}_P LC(X)$.
(2) $A \in \ddot{G}_P LC^{**}(X)$ and $B \in \ddot{G}_P LC^{*}(X) \Rightarrow A \cap B$ in $\ddot{G}_P LC(X)$.
(3) $A \in \ddot{G}_P LC^{**}(X)$ and B is open or closed $\Rightarrow A \cap B \in \ddot{G}_P LC^{**}(X)$.
(4) $A \in \ddot{G}_P LC(X)$ and B is $\ddot{g}_p$-open or $\ddot{g}_p$-closed $\Rightarrow A \cap B \in \ddot{G}_P LC(X)$.

Proof. (1) Let A in $\ddot{G}_PLC(X)$. Then there exist an $\ddot{g}_p$-open set E and $\ddot{g}_p$-closed set F such that $A = E \cap F$. So, $A \cap B = (E \cap F) \cap B = (E \cap B) \cap F$ in $\ddot{G}_PLC(X)$.

(2) Let $A = E \cap F$, where E is open and F is $\ddot{g}_p$-closed and $B = T \cap S$, where T is $\ddot{g}_p$-open and S is closed. Then $A \cap B = (E \cap F) \cap (T \cap S) = (E \cap T) \cap (F \cap S)$ where $E \cap T$ is $\ddot{g}_p$-open and $F \cap S$ is $\ddot{g}_p$-closed. Therefore, $A \cap B$ in $\ddot{G}_PLC(X)$.

(3) Let $A \in \ddot{G}_PLC^{**}(X)$. Then there exist an open set E and $\ddot{g}_p$-closed set F such that $A = E \cap F$. If B is open, then $A \cap B = (E \cap F) \cap B = (E \cap B) \cap F$ in $\ddot{G}_PLC^{**}(X)$. If B is closed which implies $A \cap B = (E \cap F) \cap B = E \cap (B \cap F)$ in $\ddot{G}_PLC^{**}(X)$.

(4) Let $A \in \ddot{G}_PLC(X)$. Then there exist an $\ddot{g}_p$-open set E and $\ddot{g}_p$-closed set F such that $A = E \cap B$. If B is $\ddot{g}_p$-open, then $A \cap B = (E \cap F) \cap B = (E \cap B) \cap F$ in $\ddot{G}_PLC(X)$. If B is $\ddot{g}_p$-closed, then $A \cap B = (E \cap F) \cap B = E \cap (B \cap F)$ in $\ddot{G}_PLC(X)$.

Theorem 3.22. *For a subset A of (X, τ) the following are equivalent:*

(1) $A \in \ddot{G}_PLC^{*}(X)$.

(2) $A = S \cap cl(A)$ for some $\ddot{g}_p$-open set S.

(3) $cl(A)/A$ is $\ddot{g}_p$-closed.

(4) $A \cup (X/cl(A))$ is $\ddot{g}_p$-open.

Proof. (1) $\Rightarrow$ (2). Let $A \in \ddot{G}_PLC^{*}(X)$. Then there exist an $\ddot{g}_p$-open set S and a closed set F in (X, τ) such that $A = S \cap F$. Since $A \subseteq S$ and $A \subseteq cl(A)$, we have $A \subseteq S \cap cl(A)$. Conversely, since $cl(A) \subseteq F, S \subseteq cl(A) \subseteq S \cap F = A$, we have that $A = S \cap cl(A)$.

(2) $\Rightarrow$ (1). Since S is $\ddot{g}_p$-open and $cl(A)$ is closed, we have $S \cap cl(A) \in \ddot{G}_PLC^{*}(X)$.

(3) $\Rightarrow$ (4). Let $G = cl(A)/A$. By assumption G is $\ddot{g}_p$-closed. $X/G = X \cap G^c = X \cap (cl(A)/A)^c = A \cap (X/cl(A))$. Since X/F is $\ddot{g}_p$-open, we have that $A \cup (X/cl(A))$ is $\ddot{g}_p$-open.

(4) $\Rightarrow$ (3). Let $G = A \cup (X/cl(A))$. By assumption G is $\ddot{g}_p$-open. Then X/G is $\ddot{g}_p$-closed. $X/G = X/(A \cup (X/cl(A)) = cl(A) \cap (X/A) = cl(A)/A, cl(A)/A$ is $\ddot{g}_p$-closed.

(4) $\Rightarrow$ (2). Let $G = A \cup (X/cl(A))$. By assumption, G is $\ddot{g}_p$-open. Now $G \cap cl(A) = A \cup (X/cl(A)) \cap cl(A) = (cl(A) \cap A) \cup (cl(A) \cap (X/cl(A))) = A \cup \phi = A$. Therefore $A = U \cap cl(A)$ for the $\ddot{g}_p$-open set G.

(2) $\Rightarrow$ (4). Let $A = S \cap cl(A)$ for some $\ddot{g}_p$-open set S. Now $A \cup (X/cl(A)) = S \cap cl(A) \cup (X/cl(A)) = S \cap (cl(A) \cup (X/cl(A))) = S \cap X = S$ is $\ddot{g}_p$-open.

Remark 3.23. *The following picture shows all the above discussions.*

$\swarrow$ **lc-set** $\searrow$

$\downarrow$

$\hat{g}$**lc-set** $\longrightarrow$ $\ddot{g}_P$**lc-set** $\longleftarrow$ $g^{\#}$**lc-set**

$\searrow$ $\quad$ $\nuparrow$ $\quad$ $\nwarrow \downarrow$

g^*s**lc-set** $\longleftarrow$ g^***lc-set**

(1) where $A \to B$ represents A implies B but not converse.

(2) where $A \nrightarrow B$ represents A and B are independent.

Theorem 3.24. *A topological space (X, τ) is $\ddot{g}_p$-sub maximal $\iff$ $\wp(X) = \ddot{G}_pLC^*(X)$.*

Proof. Necessity. Let $A \in \wp(X)$ and let $M = A \cup (cl(A))^c$. This implies that $cl(M) = cl(A) \cup (cl(A))^c = X$. Hence $cl(M) = X$. Therefore M is a dense subset of X. Since (X, τ) is $\ddot{g}_p$-sub maximal, M is $\ddot{g}_p$-open. Thus $A \cup (cl(A))^c$ is $\ddot{g}_p$-open and by Theorem 3.22, we have $A \in \ddot{G}_pLC^*(X)$.

Sufficiency. Let A be a dense subset of (X, τ). This implies $A \cup (cl(A))^c = A \cup X^c = A \cup \phi = A$. Now $A \in \ddot{G}_pLC^*(X)$ implies that $A = A \cup (cl(A))^c$ is $\ddot{g}_p$-open by Theorem 3.22. Hence (X, τ) is $\ddot{g}_p$-sub maximal.

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