APPROXIMATION FOR CERTAIN STANCU TYPE SUMMATION INTEGRAL OPERATORS

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ABSTRACT: Present paper is the study of $Q-$analogue of Szasz-Baskakov-Stancu operators. To obtain moments of these operators, we apply $Q-$derivative and $Q-$ Beta functions. We estimate some direct estimation properties for this operators.
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1. Introduction

For $x \in [0, \infty)$, the Szasz-Baskakov operators are determined as

$$G_r(f,u) = (r-1)\sum_{l=0}^{\infty} s_{r,l}(u)\int_0^{\infty} b_{r,l}(v)f(v)dv, \tag{1.1}$$

where

$$s_{r,l}(u) = \frac{e^{-ru}(ru)^l}{l!}.$$

$$b_{r,l}(v) = \left(\begin{array}{c} r+l-1 \\ l \end{array}\right)\frac{v^l}{(1+v)^{r+l}}.$$

Integral modification of Szasz-Mirakyan Baskakov operators were discussed in [9],[7],[6] etc. Q-analogue of various integral operators such as Szasz-Mirakyan operators, Szasz-Beta operators and other operators were proposed by [1],[5]. $Q-$analogue of above operators were proposed and established by [8] as follows

$$G_r^Q(f(v),u) = [r-1]_Q\sum_{l=0}^{\infty} s_{r,l}^Q(u)Q^l\int_0^{\infty/A} b_{r,l}^Q(v)f(v)d_Qv, \tag{1.2}$$

where

$$s_{r,l}^Q(u) = \frac{([r]_Qu)^l}{[l]_Q!}Q^{\frac{l(l-1)}{2}}\frac{1}{E_Q([r]_Qu)}.$$

and

$$b_{r,l}^Q(v) = \left[\begin{array}{c} r+l-1 \\ l \end{array}\right]_Q Q^{\frac{l(l-1)}{2}}\frac{v^l}{(1+v)_Q^{r+l}}.$$

For $0 \leq \alpha \leq \beta$ and $r \in N$, $0 < Q < 1$, the Stancu type generalization of operators (1.2) are given as

$$G_{r,\alpha,\beta}^Q(f(v),u) = [r-1]_Q\sum_{l=0}^{\infty} s_{r,l}^Q(u)Q^l\int_0^{\infty/A} b_{r,l}^Q(v)f\left(\frac{[r]_Qv+\alpha}{[r]_Q+\beta}\right)d_Qv, \tag{1.3}$$

where $s_{r,l}^Q(u)$ and $b_{r,l}^Q(v)$ are defined above. Here, we mention some notations of $Q-$Calculus, which can also be found in [10], [2]. For $r \in N$

$$[r]_Q = \frac{1-Q^r}{1-Q}.$$

The Q-derivative D_Qf of a function f is given by

$$(D_Qf)(w) = \frac{f(w)-f(Qw)}{(1-Q)w}, \qquad w \neq 0 \tag{1.4}$$

The Q-proper [10] integral are defined as

$$\int_0^{\infty} f(u)d_qu = (1-Q)\sum_{r=-\infty}^{\infty} f\left(\frac{Q^r}{A}\right)\frac{Q^r}{A}, \qquad A>0$$

$$and \int_0^{\infty} f(u)d_Qu = (1-Q)a\sum_{r=-\infty}^{\infty} f(aQ^r)Q^r \qquad a>0. \tag{1.5}$$

In this paper, we establish Q-analogue of Szasz-Baskakov Stancu operators (1.3). We obtain some moments and direct results in terms of modulus of continuity.

2. Auxiliary Results

This section deals with certain Lemmas.

Lemma 1. [8]

$$\begin{aligned}
(i)G_r^Q(1,u) &= 1\\
(ii)G_r^Q(v,u) &= \frac{[r]_Q}{Q^2[r-2]_Q}u+\frac{1}{Q[r-2]_Q}, \quad r>2.\\
(iii)G_r^q(v^2,u) &= \frac{[r]_Q^2}{Q^6[r-2]_Q[r-3]_Q}u^2+\frac{[r]_Q[1+Q]^2}{Q^5[r-2]_Q[r-3]_Q}u+\frac{[2]_Q}{Q^3[r-2]_Q[r-3]_Q}, \quad r>3\\
&= \frac{[r]_Q^2u^2+Q[r]_Qu[1+Q]^2+[2]_QQ^3}{Q^6[r-2]_Q[r-3]_Q}.
\end{aligned}$$

Lemma 2. *The central moments are given as*

$$G_{r,\alpha,\beta}^Q(v^m,u) = [r-1]_q\sum_{l=0}^{\infty}s_{r,l}^Q(u)Q^l\int_0^{\infty/A}b_{r,l}^Q(v)f\left(\frac{[r]_Qv+\alpha}{[r]_Q+\beta}\right)^m d_Qv$$

then

$$\begin{aligned}
(i)\mu_{r,0}^{\alpha,\beta}(u)=G_{r,\alpha,\beta}^Q(1,u) &= 1\\
(ii)\mu_{r,1}^{\alpha,\beta}(u)=G_{r,\alpha,\beta}^q(v,u) &= \frac{[r]_Q^2}{Q^2[r-2]_Q[([r]_Q+\beta))}u+\frac{([r]_Q}{Q[r-2]_Q([r]_Q+\beta)}+\frac{\alpha}{([r]_Q+\beta)}, \quad n>2\\
(iii)\mu_{r,2}^{\alpha,\beta}(u)=G_{r,\alpha,\beta}^q(v^2,u) &= \left(\frac{[r]_Q}{([r]_Q+\beta)}\right)^2\left[\frac{[r]_Q^2u^2+Q[r]_Qu[1+Q]^2+[2]_QQ^3}{Q^6[r-2]_Q[r-3]_Q}\right]\\
&\quad+\frac{2\alpha[r]_Q}{([r]_Q+\beta)^2}\left[\frac{[r]_Qu}{Q^2[r-2]_Q}+\frac{1}{Q[r-2]_Q}\right]+\left(\frac{\alpha}{([r]_Q+\beta)}\right)^2, \quad r>3.
\end{aligned}$$

Proof: Following [8], we have

$$\mu_{r,0}^{\alpha,\beta}(u)=1$$

and

$$\begin{aligned}
\mu_{r,1}^{\alpha,\beta}(u) &= [n-1]_Q\sum_{l=0}^{\infty}s_{r,l}^Q(x)Q^v\int_0^{\infty/A}b_{r,l}^Q(v)f\left(\frac{[r]_Qv+\alpha}{[r]_Q+\beta}\right)d_Qv\\
&= \frac{[r]_Q}{([r]_Q+\beta)}G_r^Q(v,u)+\frac{\alpha}{([r]_Q+\beta)}G_r^Q(1,u)\\
&= \frac{[r]_Q^2}{Q^2[r-2]_Q[([r]_Q+\beta))}u+\frac{[r]_Q}{Q[r-2]_Q([r]_Q+\beta)}+\frac{\alpha}{([r]_Q+\beta)},
\end{aligned}$$

and

$$\begin{aligned}
\mu_{r,2}^{\alpha,\beta}(u) &= [r-1]_Q\sum_{l=0}^{\infty}s_{r,l}^Q(u)Q^l\int_0^{\infty/A}b_{r,l}^Q(v)f\left(\frac{[r]_Qv+\alpha}{[r]_Q+\beta}\right)^2 d_Qv\\
&= \left(\frac{[r]_Q}{([r]_Q+\beta)}\right)^2G_r^Q(v^2,u)+\frac{2\alpha[r]_Q}{([r]_Q+\beta)^2}G_r^Q(v,u)+\left(\frac{\alpha}{([n]_q+\beta)}\right)^2S_n^q(1,x)\\
&= \left(\frac{[r]_Q}{([r]_Q+\beta)}\right)^2\left[\frac{[r]_Q^2u^2+Q[r]_Qu[1+Q]^2+[2]_QQ^3}{Q^6[r-2]_q[r-3]_Q}\right]\\
&\quad+\frac{2\alpha[r]_Q}{([r]_Q+\beta)^2}\left[\frac{[r]_Qu}{Q^2[r-2]_Q}+\frac{1}{Q[r-2]_Q}\right]+\left(\frac{\alpha}{([r]_Q+\beta)}\right)^2.
\end{aligned}$$

3. Main Results

Theorem 1. *Let $f\in E_C[0,\infty)$, then for all $Q\in E_C^2[0,\infty)$, we have*

$$\left|\overline{G}_{r,\alpha,\beta}^Q(g,u)-g(x)\right| \le \left(\delta_r(Q,u)+\gamma_r^2(Q,u)\right)\|g''\| \tag{3.1}$$

$$\overline{G}^Q_{r,\alpha,\beta}(f,u) = G^Q_{r,\alpha,\beta}(f,u)+f(u) -f\left(\frac{[r]_Q^2u}{Q^2[r-2]_Q([r]_Q+\beta)}+\frac{[r]_Q}{Q[r-2]_Q([r]_Q+\beta)}+\frac{\alpha}{([r]_Q+\beta)}\right) \tag{3.2}$$

Proof: *From (3.2), we have*

$$\overline{G}^Q_{r,\alpha,\beta}(v-u,u)=0. \tag{3.3}$$

Let $u\in[o,\infty)$ *and* $g\in E_C^2[0,\infty)$. *Using Taylor's expansion*

$$g(v)-g(u)=(v-u)g'(u)+\int_u^v(v-t)g''(v)dv$$

By (3.3), we have

$$\begin{aligned}\overline{G}^Q_{r,\alpha,\beta}(g,u)-g(u) &= \overline{G}^Q_{r,\alpha,\beta}\left((v-u)g'(u),\right)+\overline{G}^Q_{r,\alpha,\beta}\left(\int_u^v(v-t)g''(t)dt,u\right).\\ &= g'(u)\overline{G}^Q_{r,\alpha,\beta}\left((v-u),u\right)+G^Q_{r,\alpha,\beta}\left(\int_u^v(v-t)g''(t)dt,u\right)\\ &\quad-\int_u^{\frac{[r]_Q^2u+Q[r]_Q}{Q^2[r-2]_Q([r]_Q+\beta)}+\frac{\alpha}{([r]_Q+\beta)}}\left(\frac{[r]_Q^2u+Q[r]_Q}{Q^2[r-2]_Q([r]_Q+\beta)}+\frac{\alpha}{([r]_Q+\beta)}-t\right)g''(t)dt\\ &= G^Q_{r,\alpha,\beta}\left(\int_u^v(v-t)g''(t)dt,u\right)\\ &\quad-\int_u^{\frac{[r]_Q^2u+Q[r]_Q}{Q^2[r-2]_Q([r]_Q+\beta)}+\frac{\alpha}{([r]_Q+\beta)}}\left(\frac{[r]_Q^2u+Q[r]_Q}{Q^2[r-2]_Q([r]_Q+\beta)}+\frac{\alpha}{([r]_Q+\beta)}-r\right)g''(r)dr\end{aligned}$$

and

$$\begin{aligned}\left|\int_u^v(v-t)g''(t)dt\right| &\le \left|\int_u^v|(v-t)||g''(v)|dt\right|\\ &\le \|g''\|\left|\int_u^v|(v-t)|dt\right|\le(v-u)^2|g''|.\end{aligned}$$

Since

$$\begin{aligned}&\left|\int_u^{\frac{[r]_Q^2u+Q[r]_Q}{Q^2[r-2]_Q([r]_Q+\beta)}+\frac{\alpha}{([r]_Q+\beta)}}\left(\frac{[r]_Q^2u+Q[r]_Q}{Q^2[r-2]_Q([r]_QA+\beta)}+\frac{\alpha}{([r]_Q+\beta)}-t\right)g''(t)dt\right|\\ \le&\left(\frac{[r]_Q^2+Q[r]_Q}{Q^2[r-2]_Q([r]_Q+\beta)}+\frac{\alpha}{([r]_Q+\beta)}-t\right)^2\|g''\|=\gamma_r^2(Q,t)\|g''\|\end{aligned}$$

we follow that

$$\begin{aligned}\left|\overline{G'}^Q_{r,\alpha,\beta}(g,u)-g(u)\right| &= \left|\overline{G}^Q_{r,\alpha,\beta}\left(\int_u^v(v-t)g''(t)dt,u\right)\right.\\ &\quad\left.-\int_x^{\frac{[r]_q^2x+q[r]_Q}{Q^2[r-2]_Q([r]_Q+\beta)}+\frac{\alpha}{([r]_Q+\beta)}}\left(\frac{[r]_Q^2u+q[r]_Q}{Q^2[r-2]_Q([r]_Q+\beta)}+\frac{\alpha}{([r]_Q+\beta)}-t\right)g''(t)dt\right|\\ &\le G^Q_{r,\alpha,\beta}\left((v-u)^2\|g''\|,u\right)+\left(\frac{[r]_x^2+x[r]_Q}{Q^2[r-2]_Q([r]_Q+\beta)}+\frac{\alpha}{([r]_Q+\beta)}-u\right)^2\|g''\|\\ &= \delta_r(Q,u)+\gamma_r^2(Q,u)\|g''\|\end{aligned}$$

Noted that

$$G^Q_{r,\alpha,\beta}\left((v-u)^2,u\right)=\delta_r(Q,u) \tag{3.4}$$

Theorem 2. *Let* $f\in E_C[0,\infty)$, *then for every* $u\in[0,\infty)$, *there exists a constant* $L>0$ *such that*

$$\left|G^Q_{r,\alpha,\beta}(f,u)-f(u)\right| \le K\omega_2\left(f,\sqrt{\delta_r(Q,u)+\gamma_r^2(Q,u)}\right)+\omega\left(f,\gamma_r(Q,u)\right)$$

Proof. From (3.2), for $g \in E_C^2[0,\infty)$, we have

$$\begin{aligned}\left|G_{r,\alpha,\beta}^{Q}(f,u)-f(u)\right| &\leq \left|\overline{G}_{r,\alpha,\beta}^{Q}(f,u)-f(u)\right| \\ &+\left|f(u)-f\left(\frac{[r]_Q^2u+Q[n]_Q}{Q^2[r-2]_q([r]_Q+\beta)}+\frac{\alpha}{[r]_q+\beta}\right)\right| \\ &\leq \left|\overline{G}_{r,\alpha,\beta}^{Q}(f-g,u)-(f-g)(u)\right| \\ &+\left|f(u)-f\left(\frac{[r]_Q^2u+Q[r]_Q}{Q^2[r-2]_Q([r]_Q+\beta)}+\frac{\alpha}{[r]_Q+\beta}\right)\right|+\left|\overline{G}_{r,\alpha,\beta}^{Q}(Q,u)-g(u)\right| \\ &\leq \left|\overline{G}_{r,\alpha,\beta}^{Q}(f-g,u)\right|+|(f-g)(u)| \\ &+\left|f(u)-f\left(\frac{[r]_Q^2u+q[r]_Q}{Q^2[r-2]_Q([r]_Q+\beta)}+\frac{\alpha}{[r]_Q+\beta}\right)\right|+\left|\overline{G}_{r,\alpha,\beta}^{Q}(g,u)-g(u)\right|\end{aligned}$$

Taking boundedness of $\overline{G}_{r,\alpha,\beta}^{Q}$ and using (3.1), we get

$$\begin{aligned}\left|G_{r,\alpha,\beta}^{Q}(f,u)-f(u)\right| &\leq 4\|f-g\| \\ &+\left|f(u)-f\left(\frac{[r]_Q^2x+Q[r]_Q}{Q^2[r-2]_Q([r]_Q+\beta)}+\frac{\alpha}{[r]_Q+\beta}\right)\right|+\left(\delta_r(Q,u)+\gamma_r^2(Q,u)\right)\|g''\| \\ &\leq 4\|f-g\|+\left(\delta_r(g,u)+\gamma_r^2(g,x)\right)\|g''\|+\omega\left(f,\gamma_r(g,u)\right)\end{aligned}$$

Now taking infimum on the right hand side over all $g \in E_C^2[0,\infty)$ and using (3.1), we follow

$$\begin{aligned}\left|G_{r,\alpha,\beta}^{Q}(f,u)-f(u)\right| &\leq 4K_2\left(f,\delta_r(g,u)+\gamma_r^2(g,u)\right)+\omega\left(f,\gamma_r(q,u)\right) \\ &\leq 4L\left(f,\sqrt{\delta_r(g,u)+\gamma_r^2(g,u)}\right)+\omega\left(f,\gamma_r(g,u)\right) \\ &\leq K\omega_2\left(f,\sqrt{\delta_r(g,u)+\gamma_r^2(g,u)}\right)+\omega\left(f,\gamma_r(g,u)\right)\end{aligned}$$

where

$$K=4L>0.$$

□

Theorem 3. *Suppose* $\alpha \in (0,1]$ *and* R *be any bounded subset of the interval* $[0,\infty)$. *If* $f \in E_C(o,\infty)\bigcap LipM(\alpha)$, *then we have*

$$\left|G_{r,\alpha,\beta}^{Q}(f,u)-f(u)\right| \leq L\left\{\delta_r^{\frac{\alpha}{2}}(g,u)+2(d(u,G))^{\alpha}\right\}$$

where M *is a constant which depends on* α *and* $d(x,S)$ *is the distance between* x *and* S *and defined as* $d(u,R)=\inf\{(v-u): z \in R \text{ and } u \in [0,\infty)\}$ *and* $\delta_r(g,u)$ *are defined in (3.4).*

Proof: As per properties of infimum, there exists at least one point w in the closure of S, $w \in S$, such that

$$d(u,S)=|w-u|.$$

Taking in view the triangular inequality, we have

$$\begin{aligned}|f(v)-f(u)| &\leq |f(v)-f(w)|+|f(w)-f(u)| \\ \text{hence}\left|G_{r,\alpha,\beta}^{Q}(f,u)-f(u)\right| &\leq G_{r,\alpha,\beta}^{Q}\left(|f(v)-f(u)|,u\right) \\ &\leq G_{r,\alpha,\beta}^{Q}\left(|f(v)-f(w)|,u\right)+G_{r,\alpha,\beta}^{Q}\left(|f(w)-f(u)|,u\right) \\ &\leq L\left\{G_{r,\alpha,\beta}^{Q}\left(|v-w|^{\alpha},u\right)+|w-u|^{\alpha}\right\} \\ &\leq L\left\{G_{r,\alpha,\beta}^{Q}\left(|v-u|^{\alpha},u\right)+2|w-u|^{\alpha}\right\}\end{aligned}$$

we opt $l_1 = \frac{2}{\alpha}$, $l_2 = \frac{2}{2-\alpha}$ and we get $\frac{1}{l_1} + \frac{1}{l_2} = 1$, then using Holder's inequality, we have

$$\begin{aligned}\left|G^Q_{r,\alpha,\beta}(f,u) - f(u)\right| &\leq L\left\{\left[G^Q_{r,\alpha,\beta}\left(|v-u|^{\alpha l_1}, u\right)\right]^{\frac{1}{l_1}} \times \left[G^Q_{r,\alpha,\beta}\left(1^{l_2}, u\right)\right]^{\frac{1}{l_2}} + 2\,|w-u|^{\alpha}\right\}\\ &= L\left\{\left[G^Q_{r,\alpha,\beta}\left(|v-u|^2, x\right)\right]^{\frac{\alpha}{2}} + 2\,|w-u|^2\right\}\\ &= L\left\{\delta_r^{\frac{\alpha}{2}}(g,u) + 2(d(u,S))^{\alpha}\right\}.\end{aligned}$$

Hence proof is completed.

Theorem 4. *Let f be a bounded and integrable function on the interval $[0,\infty)$. For $Q = Q_r \in (0,1)$ and let second derivative of f exists at a fixed point $u \in [0,\infty)$, such that $Q_r \longrightarrow 1$ as $n \longrightarrow \infty$, then*

$$\lim_{r\longrightarrow\infty}[r]_{Q_r}\left[G^{Q_r}_{r,\alpha,\beta}(f,u) - f(u)\right] = \left[(2-\beta)u + 1 - \alpha\right] f'(u) + \left(\frac{u^2}{2} + u\right) f''(u).$$

Proof: In order to get the proof, we use Taylor's expansion

$$f(v) - f(u) = (v-u)f'(u) + (v-u)^2\left(\frac{1}{2}f''(u) + \varepsilon(v-u)\right).$$

where ε is bounded and $\lim_{z\longrightarrow 0}\varepsilon(z) = 0$. Now applying the operator $S^Q_{n,\alpha,\beta}(f)$ to the above relation, we have

$$\begin{aligned}S^Q_{r,\alpha,\beta}(f,u) - f(u) &= f'(u)S^{Q_r}_{r,\alpha,\beta}\left((v-u),u\right) + \frac{1}{2}f''(u)G^{Q_r}_{r,\alpha,\beta}\left((v-u)^2,u\right)\\ &\quad + G^{Q_r}_{r,\alpha,\beta}\left(\varepsilon(v-u)(v-u)^2,u\right)\\ &= f'(u)\gamma_r(Q_n,u) + \frac{1}{2}f''(u)\delta_r(Q_r,u) + G^{Q_r}_{r,\alpha,\beta}\left(\varepsilon(v-u)(v-u)^2,u\right)\end{aligned}$$

where $\gamma_r(Q_r,u)$ and $\delta_r(Q_r,u)$ are defined as in (3.2).
Now using Cauchy-Schwarz inequality, we have

$$[r]_{Q_r}G^{Q_r}_{r,\alpha,\beta}\left(\varepsilon(v-u)(v-u)^2,u\right) \leq \left(G^{Q_r}_{r,\alpha,\beta}\left(\varepsilon^2(v-u)\right)\right)^{\frac{1}{2}}\left([r]^2_{Q_r}G^{Q_r}_{r,\alpha,\beta}\left((v-u)^4,u\right)\right)^{\frac{1}{2}}$$

Using Lemma 1, we have

$$\lim_{r\longrightarrow\infty}[r]^2_{Q_r}S^{Q_r}_{r,\alpha,\beta}\left((v-u)^4,u\right) = 0$$

since

$$\lim_{r\longrightarrow\infty}\gamma_r(Q_r,u) = (2-\beta)u + 1 - \alpha$$

and

$$\lim_{r\longrightarrow\infty}\alpha_r(Q_r,u) = u^2 + 2u$$

This is the required result.

Theorem 5. *Let $Q = Q_r$ satisfies $0 < Q_r < 1$ and let $Q_r \to 1$ as $r \to \infty$, then for each $f \in E^*_{u^2}[0,\infty)$, we have*

$$\lim_{r\to\infty}\|S^{Qr}_{r,\alpha,\beta}(f) - f\|_{u^2} = 0$$

Proof: Using [4], we have that it is sufficient to verify the following

$$\lim_{r\to\infty}\|G^{Qr}_{r,\alpha,\beta}(v^l,u) - u^l\|_{u^2} = 0, \qquad l = 0,1,2. \tag{3.5}$$

Since $G^{Qr}_{r,\alpha,\beta}(1,u) = 1$, so in order to complete the theorem, it is suffices to show that

$$S^{Qr}_{r,\alpha,\beta}(v^l,u) = u^l \qquad l = 1,2$$

Now

$$\begin{aligned}&\|G^{Qr}_{r,\alpha,\beta}(v,u) - u\|_{u^2}\\ \leq\ &\sup_{u\in[0,\infty)}\frac{([r]^2_Q - Q^2[r-2]_Q([r]_Q+\beta)u + Q[r]_Q + \alpha Q^2[r-2]_Q}{Q^2[r-2]_Q([r]_Q+\beta)}\cdot\frac{1}{1+u^2}\\ \leq\ &\frac{([2]_Q[r]_Q - Q^2[r-2]_Q\beta) + Q[r]_Q + \alpha Q^2[r-2]_Q}{Q^2[r-2]_Q([r]_Q+\beta)}\cdot\sup_{u\in[0,\infty)}\frac{u+1}{1+u^2}\end{aligned}$$

which implies

$$\lim_{r\to\infty}\|G^{Qr}_{r,\alpha,\beta}(v,u) - u\|_{u^2} = 0$$

Finally

$$\begin{aligned}\|G^{Qr}_{r,\alpha,\beta}(v^2,u)-u^2\|_{u^2} \leq\ & \sup_{u\in[0,\infty)}\frac{u^2}{1+u^2}\left(\frac{[r]_Q}{[n]_Q+\beta}\right)^2\times\left(\frac{[r]_Q^2}{Q^6[r-2]_Q[r-3]_Q}-\left(\frac{[r]_Q+\beta}{[r]_Q}\right)^2\right)\\ &+\sup_{u\in[0,\infty)}\frac{1}{1+u^2}\left[\frac{[r]_Qu(1+Q)^2+Q^2[2]_Q}{Q^5[r-2]_Q[r-3]_Q}\right]\\ &+\sup_{u\in[0,\infty)}\frac{1}{1+u^2}\frac{2[r]_Q\alpha}{([r]_Q+\beta)}\left[\frac{[r]_Qu+Q}{Q^2[r-2]_Q}\right]+\sup_{u\in[0,\infty)}\frac{1}{1+u^2}\left(\frac{\alpha}{[r]_Q+\beta}\right)^2\\ =\ & \left(\frac{[r]_Q}{[r]_Q+\beta}\right)^2\left(\frac{[r]_Q^2}{Q^6[r-2]_Q[n-3]_Q}-\left(\frac{[r]_Q+\beta}{[r]_Q}\right)^2\right)\sup_{u\in[0,\infty)}\frac{u^2}{1+u^2}\\ &+\left[\frac{[r]_Q(1+Q)^2}{Q^5[r-2]_Q[r-3]_Q}+\frac{2[r]_Q\alpha}{([r]_Q+\beta)^2}\frac{[r]_Q}{Q^2[r-2]_Q}\right]\sup_{u\in[0,\infty)}\frac{u}{1+u^2}+\\ &\left[\left(\frac{\alpha}{[r]_Q+\beta}\right)^2+\frac{[2]_Q}{Q^3[r-2]_Q[r-3]_Q}+2[r]_Q\frac{\alpha}{([r]_Q+\beta)^2}\cdot\frac{1}{Q[r-2]_Q}\right]\sup_{u\in[0,\infty)}\frac{1}{1+u^2}\end{aligned}$$

which implies that

$$\lim_{r\to\infty}\|G^{Qr}_{r,\alpha,\beta}(v^2,u)-u^2\|_{u^2}=0$$

Hence the proof.

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