Integral Transforms of Generalized p-k-Mittag-Leffler Type Function

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Abstract: We introduced the new generalized p-k-Mittag-Leffler type function ${}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(z)$. In this paper, we establish some integral transforms of generalized p-k-Mittag-Leffler type function such as Euler-Beta transform, Fractional Fourier transform, Laplace transform and Mellin transform. The special cases of main results are also pointed briefly.

Keywords: Generalized p-k-Mittag-Leffler Type function, p-k-Pochhammer Symbol,

p-k Gamma function, Integral transform.

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1.Introduction

Recently the various generalization of Classical Mittag-Leffler Function (CMLF) [8] $E_\alpha(z)$ was given by many researchers' continuously due to applicability in applied mathematics and statistics. The generalization of $E_\alpha(z)$ was given by Wiman [2], Prabhakar [16], Shukla and Prajapati [1], Salim and Faraj [15], Khan and Ahmed [10] and many more, which known as Mittag-Leffler Type Functions or Generalized Mittag-Leffler Functions (GMLF). A brief survey with applications of the Mittag-Leffler functions is given by Haubold, Mathai and Saxena [6]. The CMLF and GMLF provides solutions of certain integral and differential equations of fractional orders. The kinetic equations, time-fractional diffusion equations, non-linear waves, statistical distribution, stochastic processes are important areas in which Mittag-Leffler functions applied. In this section we find out various integral transforms of generalized p-k-Mittag-Leffler function defined in equation (3.1).

2.Definitions

Definition 2.1 The p-k Pochhammer symbol is introduced by Gelhot [8] for $k, p \in \mathbb{R}^+$ as

$$ {}_p(\alpha)_{n,k} = \left(\frac{\alpha p}{k}\right)\left(\frac{\alpha p}{k}+p\right)\left(\frac{\alpha p}{k}+2p\right)\ldots\left(\frac{\alpha p}{k}+(n-1)p\right) \quad ;\alpha\in\mathbb{C}, n\in\mathbb{N}, \mathrm{Re}(\alpha)>0 \qquad (1)$$

or $$ {}_p(\alpha)_{n,k} = \frac{{}_p\Gamma_k(\alpha+nk)}{{}_p\Gamma_k(\alpha)} ;\alpha\in\mathbb{C}, n\in\mathbb{N}, \mathrm{Re}(\alpha)>0 \qquad (2)$$

Definition 2.2 The p-k Gamma function is defined [8] as for $k, p \in \mathbb{R}^+$ and $n \in \mathbb{N}$

$$ {}_p\Gamma_k(\alpha) = \frac{1}{k}\lim_{n\to\infty}\frac{n!\,p^{n+1}(np)^{\frac{\alpha}{k}}}{{}_p(\alpha)_{n+1,k}} \qquad \text{where } \alpha \in \mathbb{C}\setminus kZ^-,\ \operatorname{Re}(\alpha) > 0 \tag{3} $$

or $$ {}_p\Gamma_k(\alpha) = \frac{1}{k}\lim_{n\to\infty}\frac{n!\,p^{n+1}(np)^{\frac{\alpha}{k}-1}}{{}_p(\alpha)_{n,k}} \tag{4} $$

Definition 2.3 The relation of classical gamma function with p-k gamma function is given by

$$ {}_p\Gamma_k(\alpha) = \left(\frac{p}{k}\right)^{\frac{\alpha}{k}}\Gamma_k(\alpha) = \frac{p^{\frac{\alpha}{k}}}{k}\Gamma\left(\frac{\alpha}{k}\right) \quad \text{where } \alpha \in \mathbb{C}, \operatorname{Re}(\alpha) > 0\ ;\ p, k \in \mathbb{R}^+ \tag{5} $$

Definition 2.4 For $\alpha \in \mathbb{C}$; $p, k \in \mathbb{R}^+$ and $s \in \mathbb{N}$, the following identity holds [11]

$$ {}_p(\alpha)_{ns,k} = \left(\frac{p}{k}\right)^{ns}(\alpha)_{ns,k} = p^{ns}\left(\frac{\alpha}{k}\right)_{ns} \tag{6} $$

Definition 2.5 The Euler-Beta transform of the function $f(u)$; $u \in \mathbb{C}$ is defined as [7]

$$ \mathbf{B}\{f(u); x, y\} = \int_0^1 u^{x-1}(1-u)^{y-1} f(u)\,du \quad \operatorname{Re}(x) > 0, \operatorname{Re}(y) > 0 \tag{7} $$

Definition 2.6 The Wright generalized hypergeometric function is defined as [3] also called Fox-Wright function $${}_k\Psi_m\left[\begin{matrix}(\alpha_1, a_1),(\alpha_2, a_2)\ldots(\alpha_k, a_k); \\ (\beta_1, b_1),(\beta_2, b_2)\ldots(\beta_m, b_m);\end{matrix}\middle|\ z\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=0}^{k}\Gamma(\alpha_i + a_i n)}{\prod_{j=0}^{m}\Gamma(\beta_j + b_j n)}\frac{z^n}{n!} \tag{8}$$

Definition 2.7 The fractional fourier transform of ${}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(z)$ of order θ, $(0 < \theta \le 1)$ defined as [9]

$$ \psi_\theta(w) = \mathbb{F}_\theta[\psi](w) = \int_R e^{iw^{\frac{1}{\theta}}t}\psi(t)\,dt \qquad \text{where } \psi \text{ is a function belong to } \phi(R) \tag{9} $$

and for $\theta = 1$ it reduced to ordinary Fourier transform $\mathbb{F}[\phi](w) = \int_{-\infty}^{+\infty} e^{iwt}\phi(t)\,dt$ (10)

Definition 2.8 The Mellin transform of the function $f(\theta)$ is defined as [7]

$$ \mathrm{M}[f(\theta); \xi] = \int_0^\infty \theta^{\xi-1} f(\theta)\,d\theta = f^*(\xi)\ ;\ \mathbb{R}(\xi) > 0 \tag{11} $$

then $$ f(\theta) = \mathrm{M}^{-1}\left[f^*(\theta); x\right] = \frac{1}{2\pi i}\int x^{-t} f^*(t)\,dt \tag{12} $$

Definition 2.9 if f is piecewise continuous function of an exponential order from $\mathbb{R}^+$ to $\mathbb{R}$ then the Laplace transform of f defined as [13]

$$\mathbf{L}\{f(\theta);s\}=\int_0^\infty e^{-s\theta}f(\theta)\,d\theta \text{ where } \operatorname{Re}(s)>0 \tag{13}$$

3.Main Results

For $\alpha,\beta,\mu,\upsilon\in\mathbb{C}$; $\min\{\operatorname{Re}(\alpha),\operatorname{Re}(\beta),\operatorname{Re}(\mu),\operatorname{Re}(\upsilon)\}>0$ and $r,s>0$ with $s\le\operatorname{Re}(\alpha)+r$ we introduced the generalized p-k-Mittag-Leffler Type function ${}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(z)$ is expressed as

$${}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(z)=\sum_{n=0}^{\infty}\frac{{}_p(\mu)_{sn,k}}{{}_p\Gamma_k(\alpha n+\beta)}\frac{z^n}{{}_p(\upsilon)_{rn,k}} \text{ where } p,k\in\mathbb{R}^+ \tag{14}$$

Theorem 3.1 Euler-Beta transform: if the equation (14) satisfied then Euler Beta transform of generalized p-k-Mittag-Leffler function ${}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(z)$ is given by;

$$\mathbf{B}\left\{{}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(zu^\sigma);x,y\right\}=B(\sigma n+x,y)\,{}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(z) \tag{15}$$

Proof: According definition 2.5 the LHS of equation (15) is

$$\mathbf{B}\left\{{}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(zu^\sigma);x,y\right\}=\int_0^1 u^{x-1}(1-u)^{y-1}\,{}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(zu^\sigma)du$$

$$=\int_0^1 u^{x-1}(1-u)^{y-1}\sum_{n=0}^{\infty}\frac{{}_p(\mu)_{sn,k}}{{}_p\Gamma_k(\alpha n+\beta)}\frac{(zu^\sigma)^n}{{}_p(\upsilon)_{rn,k}}du$$

Interchanging the order of integral and summation we have

$$=\sum_{n=0}^{\infty}\frac{{}_p(\mu)_{sn,k}}{{}_p\Gamma_k(\alpha n+\beta)}\frac{z^n}{{}_p(\upsilon)_{rn,k}}\int_0^1 u^{\sigma n+x-1}(1-u)^{y-1}du$$

$$=B(\sigma n+x,y)\,{}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(z) \qquad \text{RHS}$$

Corollary 3.2 if the equation (14) satisfied then the following result holds true

$$\mathbf{B}\left\{{}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(zu^{\frac{\alpha}{k}});\frac{\beta}{k},\frac{\alpha}{k}\right\}=k\frac{\Gamma\left(\frac{\alpha}{k}\right)}{p^{\frac{\beta}{k}}}E^{\frac{\mu}{k},s}_{\frac{\alpha}{k},\frac{\alpha+\beta}{k},\frac{\upsilon}{k},r}(zp^{s-r-\frac{\alpha}{k}}) \tag{16}$$

Proof: if we put $\sigma=\dfrac{\alpha}{k}, x=\dfrac{\beta}{k}$ and $y=\dfrac{\alpha}{k}$ in equation (15), it reduced to

$$\mathbf{B}\left\{ {}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(zu^{\frac{\alpha}{k}}); \frac{\beta}{k}, \frac{\alpha}{k} \right\} = B\left(\frac{\alpha n+\beta}{k}, \frac{\alpha}{k}\right) {}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(z)$$

$$= \sum_{n=0}^{\infty} \frac{{}_p(\mu)_{sn,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{z^n}{{}_p(\upsilon)_{rn,k}} \frac{\Gamma\left(\frac{\alpha n+\beta}{k}\right)\Gamma\left(\frac{\alpha}{k}\right)}{\Gamma\left(\frac{\alpha n+\alpha+\beta}{k}\right)}$$

Using equation (5) and (6), we get

$$= \sum_{n=0}^{\infty} \frac{p^{sn}\left(\frac{\mu}{k}\right)_{sn}}{\frac{p^{\frac{\alpha n+\beta}{k}}}{k}\Gamma\left(\frac{\alpha n+\beta}{k}\right)} \frac{z^n}{p^{rn}\left(\frac{\upsilon}{k}\right)_{rn}} \frac{\Gamma\left(\frac{\alpha n+\beta}{k}\right)\Gamma\left(\frac{\alpha}{k}\right)}{\Gamma\left(\frac{\alpha n+\alpha+\beta}{k}\right)}$$

$$= k\frac{\Gamma\left(\frac{\alpha}{k}\right)}{p^{\frac{\beta}{k}}} \sum_{n=0}^{\infty} \frac{\left(\frac{\mu}{k}\right)_{sn}}{\Gamma\left(\frac{\alpha n+\alpha+\beta}{k}\right)} \frac{z^n p^{sn-rn-\frac{\alpha n}{k}}}{\left(\frac{\upsilon}{k}\right)_{rn}}$$

$$= k\frac{\Gamma\left(\frac{\alpha}{k}\right)}{p^{\frac{\beta}{k}}} E^{\frac{\mu}{k},s}_{\frac{\alpha}{k},\frac{\alpha+\beta}{k},\frac{\upsilon}{k},r}(zp^{s-r-\frac{\alpha}{k}}) \qquad \text{RHS}$$

Corollary 3.3 if the equation (14) satisfied then the following result holds true**;**

$$\mathbf{B}\left\{ {}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(zu^{\frac{\alpha}{k}}); \frac{\beta}{k}, \frac{\alpha}{k} \right\} == \frac{k}{p^{\frac{\beta}{k}}} {}_4\Psi_3\left[\begin{matrix} \left(\frac{\mu}{k},s\right),\left(\frac{\alpha}{k},0\right),\left(\frac{\upsilon}{k},0\right),(1,1); \\ \\ \left(\frac{\alpha+\beta}{k},\frac{\alpha}{k}\right),\left(\frac{\mu}{k},0\right),\left(\frac{\upsilon}{k},r\right); \end{matrix} \quad | \quad zp^{s-r-\frac{\alpha}{k}}\right] \tag{17}$$

Proof: According Corollary 3.2 the LHS of above equation (17)

$$= k\frac{\Gamma\left(\frac{\alpha}{k}\right)}{p^{\frac{\beta}{k}}} E^{\frac{\mu}{k},s}_{\frac{\alpha}{k},\frac{\alpha+\beta}{k},\frac{\upsilon}{k},r}(zp^{s-r-\frac{\alpha}{k}})$$

$$= k\frac{\Gamma\left(\frac{\alpha}{k}\right)}{p^{\frac{\beta}{k}}} \sum_{n=0}^{\infty} \frac{\left(\frac{\mu}{k}\right)_{sn}}{\Gamma\left(\frac{\alpha n+\alpha+\beta}{k}\right)} \frac{z^n p^{sn-rn-\frac{\alpha n}{k}}}{\left(\frac{\upsilon}{k}\right)_{rn}}$$

Using result $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$ we have

$$= k \frac{\Gamma\left(\frac{\alpha}{k}\right)}{p^{\frac{\beta}{k}}} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\mu}{k}+sn\right)\Gamma\left(\frac{\upsilon}{k}\right)}{\Gamma\left(\frac{\alpha n+\alpha+\beta}{k}\right)\Gamma\left(\frac{\mu}{k}\right)} \frac{\Gamma(n+1)}{\Gamma\left(\frac{\upsilon}{k}+rn\right)} \frac{z^n p^{sn-rn-\frac{\alpha n}{k}}}{n!}$$

$$= \frac{k}{p^{\frac{\beta}{k}}} {}_4\Psi_3 \left[\begin{matrix} \left(\frac{\mu}{k}, s\right), \left(\frac{\alpha}{k}, 0\right), \left(\frac{\upsilon}{k}, 0\right), (1,1); \\ \\ \left(\frac{\alpha+\beta}{k}, \frac{\alpha}{k}\right), \left(\frac{\mu}{k}, 0\right), \left(\frac{\upsilon}{k}, r\right); \end{matrix} \middle| \; zp^{s-r-\frac{\alpha}{k}} \right] \text{RHS}$$

Theorem 3.4 Fraction al Fourier transform (FFT): if the equation (14) satisfied then the FFT of generalized p-k-Mittag-Leffler function ${}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(z)$ is given by the following formula

$$\mathbb{F}_\theta\left[{}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(z)\right] == \frac{k}{p^{\frac{\beta}{k}}} \frac{i^{-1}}{w^{\frac{1}{\theta}}} {}_4\Psi_3 \left[\begin{matrix} \left(\frac{\mu}{k}, s\right), \left(\frac{\upsilon}{k}, 0\right), (1,1), (1,1); \\ \left(\frac{\beta}{k}, \frac{\alpha}{k}\right), \left(\frac{\mu}{k}, 0\right), \left(\frac{\upsilon}{k}, r\right); \end{matrix} \middle| \; -\frac{p^{s-r-\frac{\alpha}{k}}}{iw^{\frac{1}{\theta}}} \right] \quad (18)$$

Proof: using equation (14) in definition 2.7, we have

$$\mathbb{F}_\theta\left[{}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(z)\right] = \int_R e^{iw^{\frac{1}{\theta}}z} \sum_{n=0}^{\infty} \frac{{}_p(\mu)_{sn,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{z^n}{{}_p(\upsilon)_{rn,k}} dz$$

If we set $iw^{\frac{1}{\theta}}z = -t$,then above integral reduce as

$$= \sum_{n=0}^{\infty} \frac{{}_p(\mu)_{sn,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{1}{{}_p(\upsilon)_{rn,k}} \int_{-\infty}^{0} e^{-t} \left(\frac{-t}{iw^{\frac{1}{\theta}}}\right)^n \left(\frac{-dt}{iw^{\frac{1}{\theta}}}\right)$$

$$= \sum_{n=0}^{\infty} \frac{{}_p(\mu)_{sn,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{(-1)^{n+2}}{{}_p(\upsilon)_{rn,k}} \frac{i^{-(n+1)}}{w^{(\frac{n+1}{\theta})}} \int_0^{\infty} e^{-t} t^n dt$$

$$= \frac{i^{-1}}{w^{\frac{1}{\theta}}} \sum_{n=0}^{\infty} \frac{p^{sn}\left(\frac{\mu}{k}\right)_{sn}}{\frac{p^{\frac{\alpha n+\beta}{k}}}{k}\Gamma\left(\frac{\alpha n+\beta}{k}\right)} \frac{\Gamma(n+1)}{p^{rn}\left(\frac{\upsilon}{k}\right)_{rn}} \left(-\frac{1}{iw^{\frac{1}{\theta}}}\right)^n$$

$$=\frac{k}{p^{\frac{\beta}{k}}}\frac{i^{-1}}{w^{\frac{1}{\theta}}}\sum_{n=0}^{\infty}\frac{\left(\frac{\mu}{k}\right)_{sn}}{\Gamma\left(\frac{\alpha n+\beta}{k}\right)}\frac{\Gamma(n+1)}{\left(\frac{\upsilon}{k}\right)_{rn}}\left(-\frac{p^{s-r-\frac{\alpha}{k}}}{iw^{\frac{1}{\theta}}}\right)^{n}$$

$$=\frac{k}{p^{\frac{\beta}{k}}}\frac{i^{-1}}{w^{\frac{1}{\theta}}}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\mu}{k}+sn\right)}{\Gamma\left(\frac{\alpha n+\beta}{k}\right)\Gamma\left(\frac{\mu}{k}\right)}\frac{\Gamma\left(\frac{\upsilon}{k}\right)\Gamma(n+1)}{\Gamma\left(\frac{\upsilon}{k}+rn\right)}\frac{\Gamma(n+1)}{n!}\left(-\frac{p^{s-r-\frac{\alpha}{k}}}{iw^{\frac{1}{\theta}}}\right)^{n}$$

$$=\frac{k}{p^{\frac{\beta}{k}}}\frac{i^{-1}}{w^{\frac{1}{\theta}}}\,{}_4\Psi_3\left[\begin{matrix}\left(\frac{\mu}{k},s\right),\left(\frac{\upsilon}{k},0\right),(1,1),(1,1);\\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),\left(\frac{\mu}{k},0\right),\left(\frac{\upsilon}{k},r\right);\end{matrix}\;\middle|\;-\frac{p^{s-r-\frac{\alpha}{k}}}{iw^{\frac{1}{\theta}}}\right]\text{RHS}$$

Theorem 3.5 Laplace transform: if the equation (14) satisfied then for $\rho,\lambda,\delta\in\mathbb{C}$ and $\mathrm{Re}(\rho)>0,\mathrm{Re}(\lambda)>0,\mathrm{Re}(\delta)>0$ the following transform formula holds true

$$\int_0^{\infty}s^{\rho-1}e^{-\lambda s}\,{}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(zs^{\delta})ds=\frac{k}{\lambda^{\rho}p^{\frac{\beta}{k}}}\,{}_4\Psi_3\left[\begin{matrix}\left(\frac{\mu}{k},s\right),\left(\frac{\upsilon}{k},0\right),(1,1),(\rho,\delta),\\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),\left(\frac{\mu}{k},0\right),\left(\frac{\upsilon}{k},r\right),\end{matrix}\;\middle|\;\frac{p^{s-r-\frac{\alpha}{k}}z}{\lambda^{\delta}}\right]\tag{19}$$

Proof: using equation (14) in LHS of equation (19), we have

$$\Rightarrow\int_0^{\infty}s^{\rho-1}e^{-\lambda s}\sum_{n=0}^{\infty}\frac{{}_p(\mu)_{sn,k}}{{}_p\Gamma_k(\alpha n+\beta)}\frac{z^n s^{\delta n}}{{}_p(\mu)_{sn,k}}ds$$

$$\Rightarrow\sum_{n=0}^{\infty}\frac{{}_p(\mu)_{sn,k}}{{}_p\Gamma_k(\alpha n+\beta)}\frac{z^n}{{}_p(\mu)_{sn,k}}\int_0^{\infty}s^{\delta n+\rho-1}e^{-\lambda s}ds$$

$$\Rightarrow\sum_{n=0}^{\infty}\frac{p^{sn}\left(\frac{\mu}{k}\right)_{sn}}{\frac{p^{\frac{\alpha n+\beta}{k}}}{k}\Gamma\left(\frac{\alpha n+\beta}{k}\right)}\frac{z^n}{p^{rn}\left(\frac{\upsilon}{k}\right)_{rn}}\frac{\Gamma(n+1)}{n!}\frac{\Gamma(\rho+\delta n)}{\lambda^{\rho+\delta n}}$$

Using definition (2.6)

$$\Rightarrow\frac{k}{\lambda^{\rho}p^{\frac{\beta}{k}}}\,{}_4\Psi_3\left[\begin{matrix}\left(\frac{\mu}{k},s\right),\left(\frac{\upsilon}{k},0\right),(1,1),(\rho,\delta),\\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),\left(\frac{\mu}{k},0\right),\left(\frac{\upsilon}{k},r\right),\end{matrix}\;\middle|\;\frac{p^{s-r-\frac{\alpha}{k}}z}{\lambda^{\delta}}\right]\text{RHS}$$

Theorem 3.6 Mellin-Barnes integral formula: if the equation (14) satisfied then Mellin-Barnes integral formula of generalized p-k-Mittag-Leffler function is given as

$$ {}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(z)=\frac{kp^{-\frac{\beta}{k}}\Gamma\left(\frac{\upsilon}{k}\right)}{2\pi i\Gamma\left(\frac{\mu}{k}\right)}\int_{\Omega}\frac{\Gamma(t)\Gamma(1-t)}{\Gamma\left(\frac{\beta}{k}-\frac{\alpha t}{k}\right)}\frac{\Gamma\left(\frac{\mu}{k}-st\right)}{\Gamma\left(\frac{\upsilon}{k}-rt\right)}\left(-zp^{s-r-\frac{\alpha}{k}}\right)^{-t}dt \tag{20}$$

Where $|\arg(z)|<1$ and the contour of integration is $-i\infty$ to $+i\infty$. The poles of the integrand at $t=-n$ (to the left) from those at $t=\frac{\frac{\mu}{k}+n}{s}$ (to the right) for all $n\in\mathbb{N}\cup\{0\}$.

Proof: let the RHS of equation (20) and using the method of sum of residues at the poles $t=0,1,2,3,....$ we have

$$\Rightarrow\frac{kp^{-\frac{\beta}{k}}\Gamma\left(\frac{\upsilon}{k}\right)}{2\pi i\Gamma\left(\frac{\mu}{k}\right)}\int_{\Omega}\frac{\Gamma(t)\Gamma(1-t)}{\Gamma\left(\frac{\beta}{k}-\frac{\alpha t}{k}\right)}\frac{\Gamma\left(\frac{\mu}{k}-st\right)}{\Gamma\left(\frac{\upsilon}{k}-rt\right)}\left(-zp^{s-r-\frac{\alpha}{k}}\right)^{-t}dt$$

$$\Rightarrow\frac{kp^{-\frac{\beta}{k}}\Gamma\left(\frac{\upsilon}{k}\right)}{2\pi i\Gamma\left(\frac{\mu}{k}\right)}\sum_{n=0}^{\infty}\operatorname{Re}s(t=-n)\left[\frac{\Gamma(t)\Gamma(1-t)\Gamma\left(\frac{\mu}{k}-st\right)}{\Gamma\left(\frac{\beta}{k}-\frac{\alpha t}{k}\right)\Gamma\left(\frac{\upsilon}{k}-rt\right)}\left(-zp^{s-r-\frac{\alpha}{k}}\right)^{-t}\right]$$

$$\Rightarrow\frac{kp^{-\frac{\beta}{k}}\Gamma\left(\frac{\upsilon}{k}\right)}{\Gamma\left(\frac{\mu}{k}\right)}\sum_{n=0}^{\infty}\lim_{t\to-n}\frac{\pi(t+n)}{\sin\pi t}\frac{\Gamma\left(\frac{\mu}{k}-st\right)}{\Gamma\left(\frac{\beta}{k}-\frac{\alpha t}{k}\right)}\frac{\left(-zp^{s-r-\frac{\alpha}{k}}\right)^{-t}}{\Gamma\left(\frac{\upsilon}{k}-rt\right)}$$

$$\Rightarrow\frac{kp^{-\frac{\beta}{k}}\Gamma\left(\frac{\upsilon}{k}\right)}{\Gamma\left(\frac{\mu}{k}\right)}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma\left(\frac{\mu}{k}+sn\right)}{\Gamma\left(\frac{\alpha n+\beta}{k}\right)}\frac{\left(-zp^{s-r-\frac{\alpha}{k}}\right)^{n}}{\Gamma\left(\frac{\upsilon}{k}+rn\right)}$$

By using the equation (5) and (6)

$$\Rightarrow\sum_{n=0}^{\infty}\frac{p^{sn}\left(\frac{\mu}{k}\right)_{sn}}{\frac{p^{\frac{\alpha n+\beta}{k}}}{k}\Gamma\left(\frac{\alpha n+\beta}{k}\right)}\frac{z^n}{p^{rn}\left(\frac{\upsilon}{k}\right)_{rn}}=\sum_{n=0}^{\infty}\frac{{}_p(\mu)_{sn,k}}{{}_p\Gamma_k(\alpha n+\beta)}\frac{z^n}{{}_p(\upsilon)_{rn,k}}={}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(z)\quad\text{LHS}$$

Special case: if we put $p=k=1$ in equation (20) it reduced to the following Mellin-Barnes integral formula which obtain by Salim and Faraj [15] in 2012

$$E^{\mu,s}_{\alpha,\beta,\upsilon}(z)=\frac{\Gamma(\upsilon)}{2\pi i\Gamma(\mu)}\int_{\Omega}\frac{\Gamma(t)\Gamma(1-t)}{\Gamma(\beta-\alpha t)}\frac{\Gamma(\mu-st)}{\Gamma(\upsilon-rt)}(-z)^{-t}dt \tag{21}$$

Theorem 3.7 Mellin transform: if the equation (14) satisfied then for $s \in \mathbb{C}$ and $\operatorname{Re}(s) > 0$ the mellin transform of generalized p-k-Mittag-Leffler function is given by following formula

$$\mathbf{M}\left\{ {}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(-\theta\xi);t\right\} = \frac{kp^{-\frac{\beta}{k}}\Gamma\left(\frac{\upsilon}{k}\right)}{2\pi i\Gamma\left(\frac{\mu}{k}\right)} \frac{\Gamma(t)\Gamma(1-t)}{\Gamma\left(\frac{\beta}{k}-\frac{\alpha t}{k}\right)} \frac{\Gamma\left(\frac{\mu}{k}-st\right)}{\Gamma\left(\frac{\upsilon}{k}-rt\right)} \left(\theta p^{s-r-\frac{\alpha}{k}}\right)^{-t} \tag{22}$$

Proof: setting $z = -\theta\xi$ in equation (20), we have

$${}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(-\theta\xi) = \frac{kp^{-\frac{\beta}{k}}\Gamma\left(\frac{\upsilon}{k}\right)}{2\pi i\Gamma\left(\frac{\mu}{k}\right)} \int_\Omega \frac{\Gamma(t)\Gamma(1-t)}{\Gamma\left(\frac{\beta}{k}-\frac{\alpha t}{k}\right)} \frac{\Gamma\left(\frac{\mu}{k}-st\right)}{\Gamma\left(\frac{\upsilon}{k}-rt\right)} \left(\theta\xi p^{s-r-\frac{\alpha}{k}}\right)^{-t} dt$$

$$\Rightarrow {}_pE^{\mu,s}_{k,\alpha,\beta,\upsilon,r}(-\theta\xi) = \frac{kp^{-\frac{\beta}{k}}\Gamma\left(\frac{\upsilon}{k}\right)}{2\pi i\Gamma\left(\frac{\mu}{k}\right)} \int_\Omega \xi^{-t} f^*(t)\, dt$$

Where $f^*(t) = \frac{\Gamma(t)\Gamma(1-t)}{\Gamma\left(\frac{\beta}{k}-\frac{\alpha t}{k}\right)} \frac{\Gamma\left(\frac{\mu}{k}-st\right)}{\Gamma\left(\frac{\upsilon}{k}-rt\right)} \left(\theta p^{s-r-\frac{\alpha}{k}}\right)^{-t}$

Which is in the form of inverse Mellin-transform. So by applying the Mellin-transform both sides [equation (11) and (12)], we get proof.

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